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Relationship between the Fisher index of discrimination and the minimum test sample size

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ABSTRACT In this paper, we study the relationship between the Fisher index of discrimination of a univariate test statistic and the minimum sample size corresponding to the values of the parameters under testing, which is required to achieve predetermined probabilities of the Type I and Type II errors. We present a numerical study of the Fisher indices of discrimination of a gamma statistic and a Poisson statistic used to discriminate between the variances of a normal distribution. For fixed probabilities of the Type I and Type II errors, we show that the Fisher indices of these two statistics converge to some constant value associated with the Fisher index of a certain normal statistic, as the minimum sample size required to separate the two hypotheses goes to infinity, that is, when the two variances under testing become identical. To discriminate between two given variances of a normal distribution, approximate formulae for determining the minimum sample size required to achieve predetermined probabilities of the Type I and Type II errors are derived.

ABSTRAK Dalam kertas ini, kami mengkaji hubungan di antara indeks pembezalayan Fisher bagi satu statistik ujian univariat dan saiz sampel minimum bersepadan dengan nilai-nilai parameter di bawah pengujian yang diperlukan untuk mencapai kebarangkalian ralat Jenis I dan II yang ditentukan terlebih dahulu. Kami persembahkan satu pengajian berangka bagi indeks-indeks pembezalayan Fisher statistik gamma dan statistik Poisson yang digunakan untuk membezalayan varians-varians taburan normal. Bagi kebarangkalian ralat Jenis I dan II yang tetap, kami tunjukkan bahawa indeks Fisher dua statistik ini menumpu kepada sesuatu nilai pemalar yang bersekutu dengan indeks Fisher bagi sesuatu statistik normal, bila saiz sampel minimum yang diperlukan untuk memisahkan dua hipotesis berkenaan menuju ke infiniti iaitu, bila dua varians di bawah pengujian menjadi secaman. Bagi membezalayan dua varians taburan normal yang diberi, rumusan hampiran diterbitkan untuk menentukan saiz sampel minimum yang diperlukan untuk mencapai kebarangkalian pratentu ralat Jenis I dan II.

(Fisher index of discrimination, minimum sample size, variances of normal distribution, predetermined Type I and II errors)

INTRODUCTION

The Fisher index of discrimination has been successfully used in statistical pattern recognition to measure the separation provided by the discriminant function which has the normal or near-normal distributions under the two different hypotheses (Fukunaga, 1972). In certain composite hypothesis testing problems concerning the parameters of a random vector X, it is required to transform X by a linear transformation A so that AX is ancillary (Rao, 1973). In this case the matrix A is chosen so that the Fisher index of AX provides the maximum separation of the two hypotheses. In this paper, we study the relationship between the Fisher index of discrimination of a univariate test statistic and the minimum sample size corresponding to the values of the parameters under testing, which is required to achieve predetermined probabilities of the Type I and Type II errors.

Consider the problem of discriminating between two probability density functions (p.d.f.) belonging to the same parametric class. In testing H_0: $f = f(x|\underset{\sim}{\theta_1})$ versus H_1: $f = f(x|\underset{\sim}{\theta_2})$ using a test statistic T, the <u>*Fisher index of discrimination*</u> *of* T, denoted by ϕ_T, is defined as:

$$\phi_T = \frac{2[E_{\theta_1}(T) - E_{\theta_2}(T)]^2}{[V_{\theta_1}(T) + V_{\theta_2}(T)]}. \quad (1)$$

To illustrate this concept, consider testing H_0: $X \sim N(\mu_1, \sigma^2)$ versus H_1 : $X \sim N(\mu_2, \sigma^2)$ where $\mu_2 < \mu_1$, using the test statistic $T = \dfrac{\overline{X} - \mu_1}{\sigma / \sqrt{n}}$, where $\overline{X}$ is the mean of the random sample of size n. Then $T \sim N(0,1)$ under H_0 and $T \sim N(\sqrt{n}(\mu_2 - \mu_1)/\sigma, 1)$ under H_1. The Fisher index of T is given by

$$\phi_T = n[(\mu_1 - \mu_2)/\sigma]^2. \quad (2)$$

This index is a linear function of n. For a fixed probability of Type I error α, ϕ_T increases with the increase in sample size n, corresponding with the decrease in the probability of Type II error β. There is a minimum sample size n_{min} corresponding to a fixed β such that the probability of Type II error will be smaller than β for any sample size larger than n_{min}. We are interested in the behaviour of the function

$$\phi_T(n_{min}) = n_{min}[(\mu_1 - \mu_2)/\sigma]^2. \quad (3)$$

This paper studies $\phi_T(n_{min})$ and $\phi_Y(n_{min})$ for two test statistics T and Y used in discriminating between the variances of a normal distribution.

1. Some Properties of the Fisher Index of Discrimination

First, we present the result that the Fisher index (3) of a normal statistic is constant, depending only on the probabilities of the Type I and II errors α and β due to Tan and Yap (2002) subject to a certain condition.

Proposition 1. Suppose the test statistic T is normally distributed under H_0 and H_1, with distributions $N(\nu_1, \sigma^2)$ and $N(\nu_2, \sigma^2)$ respectively, where the critical region of the test is of the form H_0 is rejected if and only if $T \le c$ for some constant c. If α and β are the probabilities of the Type I and II errors respectively, z_α and z_β are the left-tail and right-tail percentage points of the standard normal distribution respectively defined by

$$\alpha = \int_{-\infty}^{z_\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \quad (4)$$

$$\beta = \int_{z_\beta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \quad (5)$$

then $\phi_T = (z_\beta - z_\alpha)^2$.

Proof. From the definition of α, $\alpha = P(T \le c \mid H_0) = P(Z \le (c - \nu_1)/\sigma \mid H_0)$ where $Z \sim N(0,1)$. Therefore

$$(c - \nu_1)/\sigma = z_\alpha. \quad (6)$$

Similarly, $\beta = P(Z > (c - \nu_2)/\sigma \mid H_1)$ implies that

$$(c - \nu_2)/\sigma = z_\beta. \quad (7)$$

From (6) and (7), $\phi_T = [(\nu_1 - \nu_2)/\sigma]^2 = (z_\beta - z_\alpha)^2$.

Corollary 1. In testing H_0: $X \sim N(\mu_1, \sigma^2)$ versus H_1: $X \sim N(\mu_2, \sigma^2)$, the minimum sample size n_{min} achieving a fixed probability of Type I error α and probabilities of Type II error smaller than β is the smallest integer satisfying

$$n_{min} \ge [\sigma(z_\beta - z_\alpha)/(\mu_1 - \mu_2)]^2. \quad (8)$$

Furthermore the Fisher index (3) of the test statistic T is the constant $(z_\beta - z_\alpha)^2$ provided $[\sigma(z_\beta - z_\alpha)/(\mu_1 - \mu_2)]^2$ is an integer.

Proposition 2. In testing H_0: $X \sim N(\mu_1, \sigma_1^2)$ versus H_1: $X \sim N(\mu_2, \sigma_2^2)$, where $\sigma_2^2 < \sigma_1^2$, let $T = \sum_{i=1}^{n}(X_i - \overline{X})^2/\sigma_1^2$ be the test statistic, where X_1, X_2, ... , X_n is a random sample of size n from the distribution, with sample mean $\overline{X}$. Then $T \sim$ gamma$((n-1)/2, 2)$ under H_0 and $T \sim$ gamma $((n-1)/2, 2\sigma_2^2/\sigma_1^2)$ under H_1 with

$$\phi_T = (n-1)\frac{\left[1 - (\frac{\sigma_2}{\sigma_1})^2\right]^2}{\left[1 + (\frac{\sigma_2}{\sigma_1})^4\right]}. \quad (9)$$

Proof. Under H_0, $T \sim \chi^2(n-1)$ or gamma($(n-1)/2$, 2) and $U = \sum_{i=1}^{n}(X_i-\overline{X})^2/\sigma_2^2 \sim \chi^2(n-1)$ under H_1. Since $T = (\sigma_2/\sigma_1)^2 U$, $T \sim$gamma($(n-1)/2, 2\sigma_2^2/\sigma_1^2$) under H_1.

Noting that if $T \sim$ gamma (α,β), then $E(T) = \alpha\beta$ and $V(T) = \alpha\beta^2$, we obtain

$$\phi_T = \frac{2\left\{(n-1)\left[1-(\frac{\sigma_2}{\sigma_1})^2\right]\right\}^2}{(\frac{n-1}{2})4\left[1+(\frac{\sigma_2}{\sigma_1})^4\right]}$$

$$= (n-1)\frac{\left[1-(\frac{\sigma_2}{\sigma_1})^2\right]^2}{\left[1+(\frac{\sigma_2}{\sigma_1})^4\right]}.$$

Proposition 3. In testing H_0: $X \sim N(\mu_1,\sigma_1^2)$ versus H_1: $X \sim N(\mu_2,\sigma_2^2)$, where $\sigma_2^2 < \sigma_1^2$, let $T = \sum_{i=1}^{n}(X_i-\overline{X})^2/\sigma_1^2$ be the test statistic, where $X_1, X_2, \ldots, X_n$ is a random sample of size n from the distribution, with sample mean $\overline{X}$. Furthermore, let H_0 be rejected if and only if $T \le c$ for some constant c; α and β are the probabilities of the Type I and II errors respectively. Then, if n is an odd positive integer, we have $\alpha = P[Y_1 \ge (n-1)/2]$ and $1-\beta = P[Y_2 \ge (n-1)/2]$ where $Y_1 \sim$ Poisson (m_1), $Y_2 \sim$ Poisson (m_2) with $m_1 = c/2$ and $m_2 = c\sigma_1^2/2\sigma_2^2$. Furthermore

$$\phi_Y = \frac{2(m_1-m_2)^2}{m_1+m_2}. \qquad (10)$$

Proof. From Proposition 2 and utilizing the fact that if T is gamma (a, b), where a is an integer, then $P(T \le c) = P(Y \ge a)$ where $Y \sim$ Poisson (c/b) (see for example Casella and Berger (1990), pg.101), we obtain $\alpha = P(T \le c \mid T \sim$ gamma($(n-1)/2, 2$)) $= P(Y_1 \ge (n-1)/2)$ where $Y_1 \sim$ Poisson (c/2), $1-\beta = P(T \le c \mid T \sim$ gamma($(n-1)/2, 2\sigma_2^2/\sigma_1^2$)$= P(Y_2 \ge (n-1)/2)$ where $Y_2 \sim$ Poisson $(c\sigma_1^2/2\sigma_2^2)$.

Corollary 2. The Fisher index ϕ_T given by (9) is equivalent to

$$\phi_T = (n-1)\frac{(m_2-m_1)^2}{(m_1^2+m_2^2)}, \qquad (11)$$

where m_1 and m_2 are parameters of the Poisson distribution defined in Proposition 3, namely

$$m_2/m_1 = \sigma_1^2/\sigma_2^2. \qquad (12)$$

Corollary 3. For two size-α tests with odd sample sizes n and n',

$$\phi_Y(n') > \phi_Y(n) \text{ for } n' > n. \qquad (13)$$

Remarks. Let $m_1 = c/2$ and $m_2 = c\sigma_1^2/2\sigma_2^2$ be the Poisson parameters corresponding to the test with size n and let $m_1' = c'/2$ and $m_2' = c'\sigma_1^2/2\sigma_2^2$ be the Poisson parameters corresponding to the test with size n'. Define $r' = (n'-1)/2$ and $r = (n-1)/2$ with $r' > r$. Since $\sum_{k=r'}^{\infty} e^{-m_1} m_1^k/k! < \sum_{k=r}^{\infty} e^{-m_1} m_1^k/k! = \alpha$ and $g_{r'}(m) = \sum_{k=r'}^{\infty} e^{-m} m^k/k!$ is monotonic increasing in m, $\alpha = \sum_{k=r'}^{\infty} e^{-m_1'}(m_1')^k/k!$ with $m_1' > m_1$, i.e. $c' > c$. From (10),

$$\phi_Y(n) = c\frac{\left[1-\frac{\sigma_1^2}{\sigma_2^2}\right]^2}{\left[1+\frac{\sigma_1^2}{\sigma_2^2}\right]}. \qquad (14)$$

It is clear that $\phi_Y(n') > \phi_Y(n)$ for $c' > c$.

Let α and β denote the probabilities of the Type I and II errors respectively. Given a fixed pair (α, β), we look at the sequence of tests with parameters $m_1(r)$ and $m_2(r)$ of the Poisson distribution satisfying

$$\alpha = \sum_{k=r}^{\infty} e^{-m_1} m_1^k/k!, \qquad (15)$$

$$1-\beta = \sum_{k=r}^{\infty} e^{-m_2} m_2^{\ k} / k! \,, \tag{16}$$

where $r = (n-1)/2$ and n is the odd sample size of the test. We show that the behaviour of the sequence $\{m_2(r)/m_1(r)\}$ as a function of r is crucial in determining the minimum sample size required to achieve predetermined probabilities of the Type I and II errors.

Theorem 1. Consider the hypothesis testing problem of H_0: $X \sim N(\mu_1, \sigma_1^{\ 2})$ versus H_1: $X \sim N(\mu_2, \sigma_2^{\ 2})$, where $\sigma_2^{\ 2} < \sigma_1^{\ 2}$ using the test statistic $T = \sum_{i=1}^{n} (X_i - \overline{X})^2 / \sigma_1^{\ 2}$ and a critical region of the form $T \le c$ for some constant c. If the sequence $\{m_2(r)/m_1(r)\}$ is monotonic decreasing in r for $r \ge 1$, where $m_1(r)$ and $m_2(r)$ are defined by (15) and (16) respectively, then the minimum odd sample size $n_{\min}$ of the tests with probability of the Type I error α and probabilities of Type II error not exceeding β is given by $n_{\min} = 2r^*+1$ where r^* is the smallest integer satisfying $m_2(r^*)/m_1(r^*) \le \sigma_1^{\ 2}/\sigma_2^{\ 2}$.

Proof. Given $\{m_2(r)/m_1(r)\}$ is monotonic decreasing in r for $r \ge 1$, let $r^* = (n_{\min} - 1)/2$ be the smallest integer satisfying $m_2(r^*)/m_1(r^*) \le \sigma_1^{\ 2}/\sigma_2^{\ 2}$, where $\alpha = \sum_{k=r^*}^{\infty} e^{-m_1(r^*)} (m_1(r^*))^k / k!$ and $1-\beta = \sum_{k=r^*}^{\infty} e^{-m_2(r^*)} (m_2(r^*))^k / k!$. For any size-$\alpha$ test with odd sample size $n' > n_{\min}$, let $r' = [(n'-1)/2] > r^*$ and $m_1' = m_1(r')$ be the Poisson parameter satisfying $\alpha = \sum_{k=r^*}^{\infty} e^{-m_1(r^*)} (m_1(r^*))^k / k! = \sum_{k=r'}^{\infty} e^{-m_1'} (m_1')^k / k!$. Then $m_1' > m_1(r^*)$ since $g_{r'}(m) = \sum_{k=r'}^{\infty} e^{-m} m^k / k!$ is increasing in m for a fixed r'. Furthermore the probability of Type II error β' of the test with sample size n' satisfies $1-\beta' = \sum_{k=r'}^{\infty} e^{-m_3'} (m_3')^k / k!$ where $m_3' = m_1'(\sigma_1^{\ 2}/\sigma_2^{\ 2})$. Suppose $m_2' = m_2(r')$ is the Poisson parameter satisfying $1-\beta = \sum_{k=r'}^{\infty} e^{-m_2'} (m_2')^k / k!$. Then $1-\beta < 1-\beta'$ since $g_{r'}(m) = \sum_{k=r'}^{\infty} e^{-m} m^k / k!$ is increasing in m for a fixed r' and $m_2' < m_3'$ due to $m_2'/m_1' = m_2(r')/m_1(r') < m_2(r^*)/m_1(r^*)$. In other words $\beta' < \beta$. Similarly for an odd sample size $n'' < n_{\min}$, the probability of Type II error $\beta'' > \beta$ by a similar argument.

Theorem 2. Let (α, β) be fixed probabilities and $m_1(r)$, $m_2(r)$ are the values defined by (15) and (16) respectively. Consider testing H_0: $X \sim N(\mu_1, \sigma_1^{\ 2})$ versus H_1: $X \sim N(\mu_2, \sigma_2^{\ 2})$, where $\sigma_2^{\ 2} < \sigma_1^{\ 2}$ using the statistic $T = \sum_{i=1}^{n} (X_i - \overline{X})^2 / \sigma_1^{\ 2}$ and a critical region of the form $T \le c$ for some constant c. Let $Y_1(m_1(r))$ and $Y_2(m_2(r))$ be the Poisson statistics associated with T defined by:

$$\alpha = P_{\sigma_1^2}[T \le c] = P[Y_1 \ge r], \tag{17}$$

$$1-\beta = P_{\sigma_2^2}[T \le c] = P[Y_2 \ge r]. \tag{18}$$

Then

(i) $$m_1(r) \sim r + z_\alpha r^{\frac{1}{2}} \text{ for large } r \tag{19}$$

and

$$m_2(r) \sim r + z_\beta r^{\frac{1}{2}} \text{ for large } r, \tag{20}$$

where z_α and z_β are constants defined by (4) and (5) respectively,

(ii) $$\phi_Y(r) = \frac{2[m_1(r) - m_2(r)]^2}{[m_1(r) + m_2(r)]} \to (z_\beta - z_\alpha)^2 \text{ as } r \to \infty, \tag{21}$$

(iii) $$\phi_T(r) = 2r \frac{[m_2(r) - m_1(r)]^2}{[m_1(r)^2 + m_2(r)^2]} \to (z_\beta - z_\alpha)^2 \text{ as } r \to \infty, \tag{22}$$

where $\phi_Y(r)$ and $\phi_T(r)$ are the Fisher indices of the statistics Y and T respectively.

Proof. (i) If Y has the Poisson distribution with parameter m, then $(Y-m)/\sqrt{m}$ converges in distribution to N(0, 1) as $m \to \infty$. Since

$$P[(Y_1 - m_1)/\sqrt{m_1} \geq (r-m_1)/\sqrt{m_1}\ ] = \alpha,$$
$$\lim_{m_1\to\infty} (r-m_1)/\sqrt{m_1} = -z_\alpha. \quad (23)$$

We note that $m_1(r) \to \infty$ as $r \to \infty$. Hence $m_1(r) \sim r + z_\alpha r^{\frac{1}{2}}$ for large r.

By a similar argument,

$$\lim_{m_2\to\infty} (r-m_2)/\sqrt{m_2} = -z_\beta. \quad (24)$$

Since $m_2(r) \to \infty$ as $r \to \infty$, we have $m_2(r) \sim r + z_\beta r^{\frac{1}{2}}$ for large r.

(ii) From (19) and (20), $\phi_Y(r) \sim \dfrac{2(z_\beta - z_\alpha)^2 r}{[2r + (z_\alpha + z_\beta)r^{\frac{1}{2}}]}$ for large r and hence $\phi_Y(r) \to (z_\beta - z_\alpha)^2$ as $r \to \infty$.

(iii) Similarly, from (19) and (20), $\phi_T(r) \sim \dfrac{2(z_\beta - z_\alpha)^2 r^2}{[2r^2 + 2(z_\alpha + z_\beta)r^{\frac{3}{2}} + (z_\alpha{}^2 + z_\beta{}^2)r]}$ for large r and hence $\phi_T(r) \to (z_\beta - z_\alpha)^2$ as $r \to \infty$.

Remarks. (i) It is clear from Theorem 1 that for fixed (α, β), if the sequence $\{m_2(r)/m_1(r)$ is monotonic decreasing in r for $r \geq 1$, then each $r = (n_{\min} - 1)/2$ where $n_{\min}$ is the minimum odd sample size of the tests of H$_0$: $X \sim$ N($\mu_1, \sigma_1{}^2$) versus H$_1$: $X \sim$ N($\mu_2, \sigma_2{}^2$), where $\sigma_2{}^2 < \sigma_1{}^2$, $[m_2(r)/m_1(r)] = \sigma_1{}^2/\sigma_2{}^2$ with probability of the Type I error α and probabilities of Type II error not exceeding β.

(ii) The constant $(z_\beta - z_\alpha)^2$ is the Fisher index of a certain normal statistic T defined by (3) and because there always exist constants $\mu_1, \mu_2, \sigma > 0$ such that $[\sigma(z_\beta - z_\alpha)/(\mu_1 - \mu_2)]^2$ is an integer, we can apply Corollary 1.

In Theorem 1, the minimum test sample size is obtained by studying the ratios of the Poisson parameters $\{m_2(r)/m_1(r)\}$. An analogous result can be obtained by studying the ratios of the percentage points of the chi-square distribution $\chi^2(n)$, i.e. $\{u_2(n)/u_1(n)\}$ to be defined in the sequel. Consider the tests on variances stated in Propositions 2 and 3 using the test statistic $T = \sum_{i=1}^{n}(X_i - \overline{X})^2/\sigma_1{}^2$. Noting that gamma ($\nu/2$, 2) is the distribution $\chi^2(\nu)$, we obtain

$$\alpha = P(T \leq c \mid T \sim \chi^2(n\text{-}1)), \quad (25)$$

$$1-\beta = P(T \leq c \mid T \sim \text{gamma}$$
$$((n-1)/2, 2\sigma_2{}^2/\sigma_1{}^2))$$
$$= P(T \leq c(\sigma_1{}^2/\sigma_2{}^2) \mid T \sim \chi^2(n\text{-}1)) \quad (26)$$

where H$_0$ is rejected for $T \leq c$ and $\sigma_1{}^2 > \sigma_2{}^2$.

For fixed probabilities α and β, we define the sequences $u_1(n)$ and $u_2(n)$ as follows:

$$\alpha = P(T \leq u_1(n) \mid T \sim \chi^2(n)) \quad (27)$$

$$1-\beta = P(T \leq u_2(n) \mid T \sim \chi^2(n)) \quad (28)$$

where

$$u_2(n)/u_1(n) = \sigma_1{}^2/\sigma_2{}^2. \quad (29)$$

Using a similar proof as that of Theorem 1, we can prove the following result.

Theorem 3: Consider the hypothesis testing problem H$_0$: $X \sim$ N($\mu_1, \sigma_1{}^2$) versus H$_1$: $X \sim$ N($\mu_2, \sigma_2{}^2$), where $\sigma_2{}^2 < \sigma_1{}^2$ using the test statistic $T = \sum_{i=1}^{n}(X_i - \overline{X})^2/\sigma_1{}^2$ and a critical region of the form $T \leq c$ for some constant c. If the sequence $\{u_2(n)/u_1(n)\}$ is monotonic decreasing in n for $n \geq 1$, where $u_1(n)$ and $u_2(n)$ are defined by (27) and (28) respectively, then the minimum sample size $n_{\min}$ of the tests with probability of the Type I error α and probabilities of Type II error not exceeding β is the smallest integer $n_{\min}$ satisfying $u_2(n_{\min}-1)/u_1(n_{\min}-1) \le \sigma_1^2/\sigma_2^2$.

Corollary 4. Consider the test on variances given in Theorem 1 where $\{m_2(r)/m_1(r)\}$ is monotonic decreasing in r for $r \ge 1$, $m_1(r)$ and $m_2(r)$ are defined by (15) and (16) respectively. By continuity, consider the extension of the function $h(r) = m_2(r)/m_1(r)$ to

$$h(r+1/2) = m_2(r+1/2)/m_1(r+1/2)$$
$$= u_2(2r+1)/u_1(2r+1)$$

for integers $r \ge 1$, where $u_1(2r+1)$ and, $u_2(2r+1)$ are defined by (27) and (28) respectively. Then the minimum even sample size $n_{\min}$ of the tests with probability of Type I error α and probabilities of Type II error not exceeding β is given by $n_{\min} = 2r^*+2$ where r^* is the smallest integer satisfying

$$m_2(r^*+1/2)/m_1(r^*+1/2) \le \sigma_1^2/\sigma_2^2.$$

2. Numerical Results and Approximate Formulae

For fixed (α, β), our numerical study indicates that the sequence $\{k(r)\}$ for $r \ge 1$ is monotonic decreasing in r where $k(r) = m_2(r)/m_1(r)$ and $m_1(r)$ and $m_2(r)$ are defined by (15) and (16) respectively. Table 1 lists down the values of $m_1(r)$, $m_2(r)$, $k(r)$, $\phi_T(r)$ and $\phi_Y(r)$ for selected values of r with (α, β) fixed at (0.05, 0.1). The Fisher indices $\phi_T(r)$ and $\phi_Y(r)$ are defined by (22) and (21) respectively. Since $k(r)' = \sigma_1^2/\sigma_2^2$ by (12) and the sequence $\{k(r)\}$ is monotonic decreasing in r, this is consistent with the expectation that a smaller ratio of σ_1^2/σ_2^2 would require a larger r corresponding to a larger minimum sample size $n_{\min}$ to discriminate between the hypotheses H$_0$: $X \sim \mathrm{N}(\mu_1,\sigma_1^2)$ and H$_1$: $X \sim \mathrm{N}(\mu_2,\sigma_2^2)$, where $\sigma_2^2 < \sigma_1^2$. From Table 1, we observe that the Fisher indices $\phi_T(r)$ and $\phi_Y(r)$ approach 8.6 which is close to the limit $(z_\beta - z_\alpha)^2$, namely 8.5644, as r goes to infinity. The functions $\phi_T(r)$ and $\phi_Y(r)$ are displayed in Figure 1. Given (α, β), suppose that we wish to determine the minimum odd sample size $n_{\min}$ of the tests of H$_0$: $X \sim \mathrm{N}(\mu_1,\sigma_1^2)$ versus H$_1$: $X \sim \mathrm{N}(\mu_2,\sigma_2^2)$, where $\sigma_2^2 < \sigma_1^2$, with probability of the Type I error α and probabilities of Type II error not exceeding β. This is equivalent to given a certain value of $k(r) = \sigma_1^2/\sigma_2^2$, we are required to find the value of r corresponding to this $k(r)$ and then $n_{\min} = 2r+1$ if r is an integer. We shall discuss two approximate formulae for finding the value of r corresponding to $k(r)$.

From (19) and (20), we can assume the approximation:

$$k(r) = m_2(r)/m_1(r)$$
$$= (r_A + z_\beta r_A^{\frac{1}{2}})/(r_A + z_\alpha r_A^{\frac{1}{2}}). \qquad (30)$$

Solving (30) for r_A, we obtain

$$r_A = [(z_\beta - kz_\alpha)/(k-1)]^2. \qquad (31)$$

The approximation (30) is more accurate for larger values of r or smaller values of $k(r)$ (close to 1).

We consider another approximation

$$k(r) = m_2(r)/m_1(r) =$$
$$(r_C + z_\beta r_C^{\frac{1}{2}} + \varsigma_\beta)/(r_C + z_\alpha r_C^{\frac{1}{2}} + \varsigma_\alpha) \qquad (32)$$

where the constants ς_α and ς_β depend only on α and β respectively. Solving the equation

$$(k-1)r_C + (kz_\alpha - z_\beta)r_C^{\frac{1}{2}} + (k\varsigma_\alpha - \varsigma_\beta) = 0 \qquad (33)$$

for r_C, we obtain

$$r_C = \left[[(z_\beta - kz_\alpha) + \sqrt{(z_\beta - kz_\alpha)^2 - 4(k-1)(k\varsigma_\alpha - \varsigma_\beta)}]/2(k-1)\right]^2 \qquad (34)$$

Define r_A' as the smallest integer larger than or equal to r_A and r_C' as the smallest integer larger than or equal to r_C.

For given values of $k(r)$ in Table 1 from 4.07142857 to 1.029758661, the estimated values of r by r_A' from (31) and r_C' from (34) are given in Table 2, together with the absolute and percentage errors in estimating r, where $\alpha = 0.05$ and $\beta = 0.1$. We note that to use the formula (34), we need to estimate the constants ς_α and ς_β from known values of r, $m_1(r)$ and $m_2(r)$. The actual values of $\varsigma_\alpha(r) = m_1(r) - r - z_\alpha r^{\frac{1}{2}}$ and $\varsigma_\beta(r) = m_2(r) - r - z_\beta r^{\frac{1}{2}}$ depend on r. In this example, the actual values $\varsigma_\alpha(r)$ and $\varsigma_\beta(r)$ are calculated for r = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000. The constants ς_α and ς_β used in (34) are the averages of the 10 calculated values of $\varsigma_\alpha(r)$ and $\varsigma_\beta(r)$ respectively. These are namely $\varsigma_\alpha = 0.375$ and $\varsigma_\beta = 0.074$.

We observe that the absolute error in estimating the minimum odd sample size $n_{\min}$ by (31) or (34) is double the absolute error in estimating r. The percentage errors in estimating $n_{\min}$, namely $\frac{|r - r_A'|}{(r+\frac{1}{2})} \text{x}100$ and $\frac{|r - r_C'|}{(r+\frac{1}{2})} \text{x}100$ are roughly the same as the percentage errors in estimating r. From Table 2, we observe that the absolute error in estimating r by r_A' varies from 2 to 7 with the corresponding percentage errors varying from 40% to 0.5%. The percentage error is smaller for smaller values of k close to 1 corresponding to larger values of r. On the other hand, the absolute error in estimating r with r_C' varies from 0 to 2 with the corresponding percentage errors varying from 20% to 0%. We conclude that the estimator r_C is better than r_A. However, the estimator r_A is easier to use compared with r_C because to use r_C, we need to estimate the constants ς_α and ς_β from calculated values of $\varsigma_\alpha(r)$ and $\varsigma_\beta(r)$ for a set of values r.

Table 1. Values of $m_1(r)$, $m_2(r)$, $k(r)$, $\phi_T(r)$ and $\phi_Y(r)$ for selected values of r with (α, β) fixed at (0.05, 0.1).

	α	$\beta = 0.1$			
r	0.05	1- $\beta = 0.9$			
	m_1	m_2	$k = m_2/m_1$	ϕ_T	ϕ_Y
5	1.96	7.98	4.07142857	5.36719884	7.29183099
10	5.41	14.19	2.62292052	6.68519609	7.86616327
15	9.22	20.11	2.18112798	7.26933792	8.08674395
20	13.23	25.88	1.95616024	7.57675745	8.18320123
25	17.35	31.56	1.81902017	7.78393857	8.25696586
30	21.56	37.17	1.72402597	7.91810522	8.29804529
35	25.83	42.73	1.65427797	8.01940024	8.33168028
40	30.15	48.25	1.60033167	8.09642008	8.35739796
45	34.51	53.74	1.55722979	8.15934075	8.38057564
50	38.91	59.21	1.52171678	8.20927126	8.39971464
60	47.79	70.07	1.46620632	8.28056481	8.42352622
70	56.76	80.87	1.42477097	8.33679311	8.44717140
80	65.81	91.60	1.39188573	8.36534381	8.45084937
90	74.91	102.30	1.36563877	8.39956919	8.46692737
100	84.06	112.95	1.34368308	8.42051156	8.47299223
200	177.20	218.24	1.23160271	8.52492859	8.51851912
300	271.94	322.30	1.18518791	8.55698859	8.53570813
400	367.50	425.72	1.15842177	8.57321576	8.54635133

	α	$\beta = 0.1$			
r	0.05	1- $\beta = 0.9$			
	m_1	m_2	$k = m_2/m_1$	ϕ_T	ϕ_Y
500	463.59	528.74	1.140533661	8.583781177	8.554659236
600	560.06	631.46	1.127486341	8.587135419	8.557069961
700	656.81	733.96	1.117461671	8.589836316	8.559463463
800	753.79	836.29	1.109446928	8.591171508	8.560890018
900	850.95	938.48	1.102861508	8.593083729	8.563062987
1000	948.27	1040.55	1.097314056	8.593106956	8.563468187
1500	1436.51	1549.62	1.078739445	8.596331691	8.568864785
2000	1926.59	2057.27	1.067829689	8.598705106	8.573224159
2500	2417.86	2564	1.060441878	8.597686691	8.573865825
3000	2909.96	3070.09	1.055028248	8.598164177	8.575719902
3500	3402.7	3575.69	1.05083904	8.597907349	8.576631601
4000	3895.94	4080.9	1.047475064	8.597663065	8.577381921
5000	4883.59	5090.43	1.042354088	8.597493419	8.578844959
6000	5872.42	6099.04	1.038590564	8.597216725	8.579843127
7000	6862.15	7106.96	1.035675408	8.596971772	8.580637721
8000	7852.59	8114.33	1.033331678	8.596688461	8.581220123
9000	8843.61	9121.25	1.031394419	8.596371612	8.581638777
10000	9835.12	10127.8	1.029758661	8.59616236	8.582069397

Table 2. Estimated values of r, namely $r_A^{'}$ and $r_C^{'}$ for given values of $k(r)$ together with the absolute and percentage errors in estimating r, where $\alpha = 0.05$ and $\beta = 0.1$.

k	r (actual)	r'A	|r - r'A|	|r - r'A|/r x100%	r'C	|r - r'C|	|r - r'C|/r x100%
4.0714285714	5	7	2	40.00	6	1	20.00
2.6229205176	10	12	2	20.00	11	1	10.00
2.1811279826	15	17	2	13.33	16	1	6.67
1.9561602419	20	23	3	15.00	21	1	5.00
1.8190201729	25	28	3	12.00	26	1	4.00
1.7240259740	30	33	3	10.00	31	1	3.33
1.6542779714	35	38	3	8.57	36	1	2.86
1.6003316750	40	43	3	7.50	41	1	2.50
1.5572297885	45	48	3	6.67	46	1	2.22
1.5217167823	50	53	3	6.00	51	1	2.00
1.4662063193	60	63	3	5.00	61	1	1.67
1.4247709655	70	73	3	4.29	71	1	1.43
1.3918857317	80	84	4	5.00	81	1	1.25
1.3656387665	90	94	4	4.44	91	1	1.11
1.3436830835	100	104	4	4.00	101	1	1.00
1.2316027088	200	204	4	2.00	201	1	0.50
1.1851879091	300	305	5	1.67	301	1	0.33
1.1584217687	400	405	5	1.25	401	1	0.25
1.1405336612	500	505	5	1.00	500	0	0.00
1.1274863407	600	606	6	1.00	600	0	0.00
1.1174616708	700	706	6	0.86	700	0	0.00
1.1094469282	800	806	6	0.75	800	0	0.00
1.1028615077	900	906	6	0.67	900	0	0.00
1.0973140561	1000	1007	7	0.70	1000	0	0.00
1.0787394449	1500	1507	7	0.47	1498	2	0.13

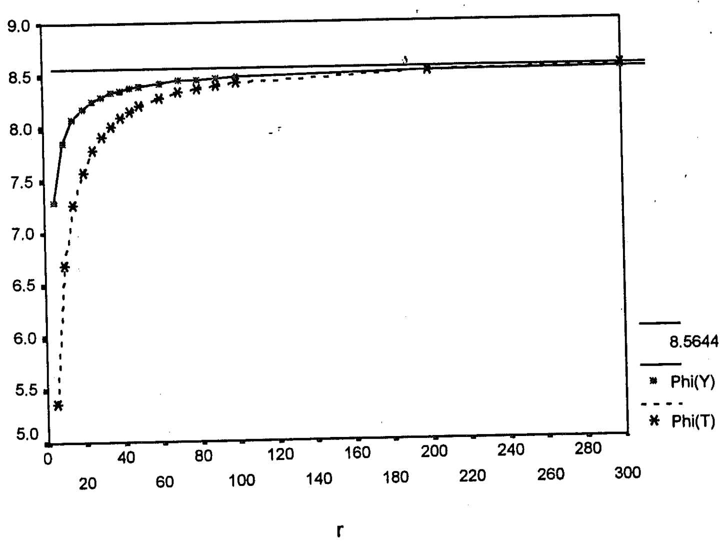

Figure 1. Graphs of the Fisher indices $\phi_T(r)$ and $\phi_Y(r)$ for $\alpha = 0.05$ and $\beta = 0.1$.

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