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CONSTRUCTING QUANTUM ANGULAR MOMENTUM $\hat{L}_3$ IN A SPECIFIC DIRECTION BY USING THE $\mathrm{U}(1)$ GROUP

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Abstract: The purpose of this work is to investigate the mathematical structure of finite quantum angular momentum in a specific direction L_3 which can be constructed from the representation of the $\mathrm{U}(1)$ group. The angular momentum eigenstate is invariant under Abelian rotation symmetry. The character group of $\mathrm{U}(1)$ is constructed to show that there exists an additive unitary operator for the angular momentum eigenstate, and the rotation eigenstate is invariant under it. The Weyl relation is proved by showing that the angle $e^{i\phi}$ and angular momentum L_3 are the canonical conjugate pair of observables.

Keywords: *group, character group, group representation, quantum angular momentum.*

1. Introduction

Traditionally, a quantisation scheme, such as the transcription of the Lie algebra structure from phase space to Hilbert space has been used to represent the observables of a physical system as self-adjoint operators on Hilbert space. We refer to Busch et al., (1997), Dirac, (1981), Isham, (1984), Sundermeyer, (1982) as reading materials to provide some methods of quantisation. The physical system of interest in this work is a quantum system with Abelian rotation symmetry. The reason is because angular momentum is a conserved quantity when the system has rotational symmetry (Hall, 2013). The discussion of angle and angular momentum observables has appeared in the past decades (Berry, 1977; Bizarro, 1994; Yamada, 1982) but with different spirit and objectives. They used Weyl-Wigner formulation to study rotation-angle and angular momentum in the quantum system. Recent developments in this technique can be seen in Weinbub and Ferry, (2018). In this work, the role of $\mathrm{U}(1)$ group and its character group are analysed to formulate the quantum angular momentum $\hat{L}_3$ in specific direction.

The $\mathrm{U}(1)$ group has been used to study various kinds of physical systems. For instance, in quantum harmonic oscillator. It has $\mathrm{U}(1)$ symmetry which is rotations in the position-momentum plane (Iwai, 1982). This implies that the Hamiltonian's eigenvalues, which represent the system's energy, will be integers multiplied by some fixed value. In addition, $\mathrm{U}(1)$ can act on complex-valued wavefunctions by pointwise phase transformations to describe quantum particles (Das, 2020). This technique is used to study how particles interact with electromagnetic fields, and the electric charge of the state is the physical interpretation of the operator's eigenvalue. For the recent discussion on this area see Zhang and Feng, (2023). For the case of quantum angular momentum, since the group $\mathrm{SO}(2)$ is the circle group for rotations in $\mathbb{R}^2$ plane and it is isomorphic to $\mathrm{U}(1)$ (Woit et al., 2017) then the group $\mathrm{U}(1)$ can be seen as a group of rotation in complex plane. The eigenvalues of the generator $\hat{L}_3$ are integers that are not continuous, implying that the quantum angular momentum on specific direction has different behaviour compared to classical angular momentum.

The paper is structured as follows. Section 2 focuses on constructing angular momentum $\hat{L}_3$ by applying the representation of $\mathrm{U}(1)$ group and shows that the angular momentum is invariant under rotation symmetry. In Section 3 the character group of $\mathrm{U}(1)$ is used to construct the additive unitary group of shift operator for angular momentum and prove that the angle eigenstate is invariant under these operators. Section 4 is devoted to proving the Abelian rotation symmetry and additive unitary shift operator for angular momentum satisfy the Weyl canonical commutation relation. In the final section, the conclusion of this work is stated.

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2. The Representation of $\mathrm{U}(1)$ Group

In the three-dimensional space, if one chooses a particular direction, then the group of rotation on that particular axis can be identified with the abelian group $\mathrm{U}(1)$.

Definition 1. The $\mathrm{U}(1)$ group

- The elements of the group are points on the unit circle which are denoted by a unit complex number $e^{i\phi}$, or an angle $\phi \in \mathbb{R}$ with ϕ and $\phi + 2\pi k$ denoting the same group element for $k \in \mathbb{Z}$.
- Multiplication of group elements is complex multiplication:
$$e^{i\phi_1}e^{i\phi_2} = e^{i(\phi_1+\phi_2)},$$
where $\phi_1, \phi_2 \in \mathbb{R}$. So, in terms of angles the group law is addition (mod 2π)
- It is an Abelian group.

Definition 2. An irreducible representation of $\mathrm{U}(1)$ is given by,
$$U_k : U(1) \to GL(1, \mathbb{C}) \cong \mathbb{C}^\times$$
$$: e^{i\phi} \mapsto e^{ik\phi}$$
where $k \in \mathbb{Z}$ and $\mathbb{C}^\times$ is an invertible group of complex numbers. The map U_k has some properties:

- It satisfies the homomorphism rules $U_k(\phi_1 + \phi_2) = U_k(\phi_1)U_k(\phi_2)$.
- It has periodicity property: $U_k(2\pi) = U_k(0) = 1$.
- It has unitary property: $\overline{e^{ik\phi}}e^{ik\phi} = 1 = e^{ik\phi}\,\overline{e^{ik\phi}}$.

Any unitary representation U can be written as a direct sum,
$$U = U_1 \oplus U_2 \oplus \ldots \oplus U_n , \tag{1}$$
where U_i for $i = 1, 2, \ldots, n$ are the irreducible representation. Since $\mathrm{U}(1)$ is commutative, all irreducible representations will be one dimensional. Now, we can write a unitary representation of U in terms of its irreducible representation as the following:
$$U(e^{i\phi}) = \begin{pmatrix} e^{ik_1\phi} & \ldots & \ldots & 0 \\ 0 & e^{ik_2\phi} & \ldots & 0 \\ \ldots & \ldots & & \ldots \\ 0 & 0 & \ldots & e^{ik_n\phi} \end{pmatrix} = e^{i\hat{L}_3\phi} . \tag{2}$$
Hence, we can define the generator $\hat{L}_3$ as the following,
$$\hat{L}_3 = \begin{pmatrix} k_1 & \ldots & \ldots & 0 \\ 0 & k_2 & \ldots & 0 \\ \ldots & \ldots & & \ldots \\ 0 & 0 & \ldots & k_n \end{pmatrix}. \tag{3}$$
If we let $U(\phi) = e^{i\hat{L}_3\phi}$, and differentiate it with respect to ϕ, we get the following equation,
$$-i\frac{d}{d\phi}U(\phi) = \hat{L}_3 U(\phi) . \tag{4}$$
This implies the generator $\hat{L}_3 = -i\frac{d}{d\phi}$. If $\psi\,(0)$ is the unit vector that describes our system at $\phi = 0$, then the unit vector which describes our system at angle ϕ is given by,
$$\psi(\phi) = U(\phi)\psi(0) . \tag{5}$$
By substituting $\psi(0)$ into both sides of equation (4), then we obtain the following equation:
$$-i\frac{d}{d\phi}\psi(\phi) = \hat{L}_3\psi(\phi) . \tag{6}$$
The equation above is exactly the equation for quantum angular momentum on a specific z-axis. The operator $\hat{L}_3$ is called the angular momentum operator on the z-axis and it governs the quantum system with Abelian rotation symmetry.

Observe that $U(\phi)$ can be written as the following,
$$U(\phi) = e^{i\hat{L}_3\phi} = \sum_{k=1}^{n} e^{ik\phi} \; |k\rangle\langle k| , \tag{7}$$
where $|k\rangle$ is the eigenbasis for $\widehat{L_3}$. By substituting the eigenbasis $|k\rangle$ into the right-hand side of equation (4) gives,
$$\begin{aligned} \hat{L}_3(U(\phi)|k\rangle) &= e^{ik\phi}(\hat{L}_3|k\rangle) , \\ &= ke^{ik\phi}|k\rangle , \\ &= U(\phi)\hat{L}_3|k\rangle . \end{aligned}$$
Therefore, the unitary operator $U(\phi)$ and the angular momentum generator $\hat{L}_3$ commute,
$$\left[U(\phi), \hat{L}_3\right] = 0 . \tag{8}$$
Physically, this implies that if the state has well-defined eigenvalue of $\hat{L}_3$ at angle zero, then it will continue to have the same value at any other angle ϕ. In other words, the angular momentum is invariant under rotation symmetry.

3. Character Group of $\mathrm{U}(1)$

A character for any Abelian group G is a homomorphism $\chi : \mathrm{G} \to \mathbb{C}^\times$, where $\mathbb{C}^\times$ is a group of invertible complex numbers. The character forms an Abelian group under multiplication. The group is called dual group and denoted as G^. When U_k is a irreducible unitary representation then the character corresponds to,
$$\chi_k(\phi) = \mathrm{Tr}\big(U_k(\phi)\big) . \tag{9}$$
This equation implies that,
$$\chi_k(\phi) = \mathrm{Tr}\big(e^{ik\phi}\big) = e^{ik\phi} = U_k(\phi) . \tag{10}$$
So, the irreducible unitary representation of the $\mathrm{U}(1)$ can be specified by its character. Physically, it means if we interpret χ_k as particle moving with angular momentum k on the circle and $\chi_{\delta k}$ as moving with angular momentum δk, then $\chi_k \cdot \chi_{\delta k}$ should be interpreted as moving with angular momentum $k + \delta k$.

Recall that, by solving equation (6), the eigenvector $|k\rangle$ is given as $\psi_k(\phi) = e^{ik\phi}$ where $k \in \mathbb{Z}$. This is exactly the characters $\chi_k(\phi)$. Since the characters form a group G^, then we can introduce an additive unitary group of shift operator:
$$V(k) = e^{ik\hat{\Phi}} = \sum_{\phi} e^{ik\phi} \; |\phi\rangle\langle\phi| , \tag{11}$$
where $|\phi\rangle$ is the eigenvector for operator $V(k)$. By using the Fourier transform the eigenvector $|\phi\rangle$ can be written in terms of $|\mathrm{m}\rangle$ as,
$$|\phi\rangle = \frac{1}{\sqrt{2\pi}}\sum_{m} e^{ik\phi}\,|m\rangle . \tag{12}$$
Thus,
$$V(k)|\phi\rangle = e^{ik\phi}|\phi\rangle . \tag{13}$$
This shows that the improper eigenvectors are invariant under additive unitary group of shift operator $V(k)$.

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4. Weyl Canonical Commutation Relation for Angle and Angular Momentum

In quantum mechanics, canonical commutation relation (CCR) of observables plays a special role as it relates the fundamental relation between observables. For example, position and momentum. The observables which form the CCR are called canonically conjugate pair of observables and they imply the Heisenberg uncertainty principle. There is another way of writing it by using exponential and it is called Weyl canonical commutation relation.

Recall that, the unitary group of operators for rotation and additive unitary shift operator angular momentum are given by $U(\phi) = e^{i\hat{L}_3\phi}$ and $V(k) = e^{ik\hat{\Phi}}$ respectively. Those act as $U(\phi)\,|\phi'\rangle = |\phi + \phi'\rangle$ and $V(k)|\phi'\rangle = e^{ik\phi}|\phi'\rangle$. Thus for any state $|\phi'\rangle$:

$$U(\phi)V(k)|\phi'\rangle = e^{ik\phi'}U(\phi)|\phi'\rangle = e^{ik\phi'}|\phi + \phi'\rangle\,,$$

$$V(k)U(\phi)|\phi'\rangle = V(k)|\phi + \phi'\rangle = e^{ik(\phi+\phi')}|\phi + \phi'\rangle\,,$$

Hence, we find the relation,

$$V(k)U(\phi) = e^{ik\phi}U(\phi)V(k)\,. \quad (14)$$

This implies that angular momentum and angle are the pair of canonically conjugate self-adjoint operators.

5. Conclusion

In this work, the finite quantum angular momentum in a specific direction, $\hat{L}_3$ can be constructed by the unitary representation of the $\mathrm{U}(1)$ group. The eigenstate of $\hat{L}_3$ is invariant under rotation symmetry. We have constructed the character of $\mathrm{U}(1)$ group and shown that the angular momentum eigenstate is invariant under the additive unitary operator. The finding has also shown that the angle $e^{i\phi}$ and the angular momentum $\hat{L}_3$ are the canonical conjugate observables. This is because they satisfied the Weyl canonical commutation relation.

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