A new beta generated Kumaraswamy Marshall-Olkin-G family of distributions with applications

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ABSTRACT Unification of the recently introduced Kumaraswamy Marshall-Olkin-G and Beta Marshall-Olkin-G family of distributions is proposed. A number of important statistical and mathematical properties of the family is investigated. A distribution belonging to the proposed family is shown to perform better than the corresponding distribution from the Kumaraswamy Marshall-Olkin-G and Beta Marshall-Olkin-G family of distributions by considering data fitting with three real life data sets.

Key words: Exponentiated family, Power Weighted Moments, AIC and K-S test.

INTRODUCTION

Efforts to define new families of continuous distributions by extending well-known distributions to provide greater flexibility in modelling different types of data generated from real life situation has received renewed attention of many researchers. To this end several new classes were proposed by adding one or more parameters.

Some important recent contributions in this area include Marshall-Olkin Kumaraswamy-G family introduced by Handique et al., (2017a), Generalized Marshall-Olkin Kumaraswamy-G family (Chakraborty and Handique, 2017a), beta generated Kumaraswamy-G (Handique et al., 2017b), Kumaraswamy Generalized Marshall-Olkin-G family (Chakraborty and Handique, 2017b) and beta Generalized Marshall-Olkin-G family (Handique and Chakraborty, 2016) among others. Alizadeh et al., (2015a) proposed Kumaraswamy Marshall-Olkin-G ($\mathrm{KwMO-G}$) family of distributions as a new extension of the Marshall-Olkin (MO) family for a given baseline distribution with cumulative distribution function (cdf) $G(t)$, survival function (sf) $\overline{G}(t)=1-G(t)$ and probability density function (pdf) $g(t)$. The cdf and sf of the $\mathrm{KwMO-G}$ are given by

$$F^{\mathrm{KwMOG}}(t;a,b,\alpha) = 1-[1-[G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^{a}]^{b} \quad (1)$$

and

$$\overline{F}^{\mathrm{KwMOG}}(t;a,b,\alpha) = [1-[G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^{a}]^{b} \quad (2)$$

where $\alpha>0\,(\alpha+\overline{\alpha}=1)$ and $a>0, b>0$ are two additional shape parameters. The density function corresponding to (1) is given by

$$f^{\mathrm{KwMOG}}(t;a,b,\alpha) = \frac{a\,b\,\alpha\, g(t)\,G(t)^{a-1}}{[1-\overline{\alpha}\,\overline{G}(t)]^{a+1}}\left[1-\left\{\frac{G(t)}{1-\overline{\alpha}\,\overline{G}(t)}\right\}^{a}\right]^{b-1}$$

(3) $-\infty<t<\infty;\ \alpha>0; a>0, b>0$ Alizadeh et al., (2015b) proposed another family of distributions called Beta Marshall-Olkin-G ($\mathrm{BMO-G}$) family of distributions as a new extension of the Marshall-Olkin (MO)

family for a given baseline distribution with cdf $G(t)$, sf $\overline{G}(t)=1-G(t)$ and pdf $g(t)$. The cdf, sf and pdf of the $\mathrm{BMO-G}$ are given respectively by

$F^{\mathrm{BMOG}}(t;m,n,\alpha)=I_{\frac{G(t)}{1-\overline{\alpha}\,\overline{G}(t)}}(m,n)$,

$\overline{F}^{\mathrm{BMOG}}(t;m,n,\alpha)=1-I_{\frac{G(t)}{1-\overline{\alpha}\,\overline{G}(t)}}(m,n)$ and

$$f^{\mathrm{BMOG}}(t;m,n,\alpha)=\frac{(1/B(m,n))\,\{\alpha^{n}\,g(t)G(t)^{m-1}\overline{G}(t)^{n-1}\}}{[1-\overline{\alpha}\,\overline{G}(t)]^{m+n}}$$

where $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$ and

$I_t(m,n)=B(m,n)^{-1}\int_0^t x^{m-1}(1-x)^{n-1}dx$ denotes the incomplete beta function ratio. $\alpha>0$ $(\alpha=1-\overline{\alpha})$ and $m>0, n>0$ are two additional shape parameters.

For both these families the authors have studied many statistical and mathematical properties and have shown that distributions belonging to their families provide better fit than some competing models in fitting real data sets.

The main motivation of this article is to provide a new family that unifies $\mathrm{KwMO-G}$ and $\mathrm{BMO-G}$ families in to one parent family by using the beta generated technique of Eugene et al. (2002), investigate some of its important statistical and mathematical properties and carry out comparative data modelling applications.

Beta Kumaraswamy Marshall-Olkin-G ($BKwMO-G$) family of distributions

Here a beta generated $\mathrm{KwMO-G}$ $(\mathrm{BKwMO-G})$ family is introduced with pdf, cdf and hazard rate function (hrf) respectively are given by

$$f^{\mathrm{BKwMOG}}(t;m,n,a,b,\alpha)=\frac{a\,b\,\alpha\,g(t)\,G(t)^{a-1}}{B(m,n)\,[1-\overline{\alpha}\,\overline{G}(t)]^{a+1}}\left[1-\left\{\frac{G(t)}{1-\overline{\alpha}\,\overline{G}(t)}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{\frac{G(t)}{1-\overline{\alpha}\,\overline{G}(t)}\right\}^{a}\right]^{b}\right]^{m-1} \quad (4)$$

$$0<t<\infty,\ 0<a,b<\infty, m,n>0,\ \alpha>0$$

$$F^{\mathrm{BKwMOG}}(t;m,n,a,b,\alpha)=I_{1-[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^{a}\,]^{b}}(m,n) \quad (5)$$

and $h^{\mathrm{BKwMOG}}(t;m,n,a,b,\alpha)$

$$=\frac{a\,b\,\alpha\,g(t)\,G(t)^{a-1}}{B(m,n)\,[1-\overline{\alpha}\,\overline{G}(t)]^{a+1}[1-I_{1-[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^{a}]^{b}}(m,n)]}\{1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^{a}\}^{bn-1}$$
$$\times\{1-[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^{a}\,]^{b}\}^{m-1} \quad (6)$$

Important particular cases

For

(i) $m=n=1$, $\mathrm{BKwMO\text{-}G}(m,n,a,b,\alpha)\equiv\mathrm{KwMO\text{-}G}(a,b,\alpha)$ (Alizadeh et al., 2015a), (ii) $a=b=1$, $\mathrm{BKwMO\text{-}G}(m,n,a,b,\alpha)\equiv\mathrm{BMO\text{-}G}(m,n,\alpha)$ (Alizadeh et al., 2015b), (iii) $\alpha=1$, $\mathrm{BKwMO\text{-}G}(m,n,a,b,\alpha)=\mathrm{BKw}(m,n,a,b)$ (Handique et al., 2017b).

(iv) $m=n=a=b=1$, $\mathrm{BKwMO-G}\ (m,n,a,b,\alpha)\equiv\mathrm{MO}(\alpha)$ (Marshall and Olkin, 1997),

(v) $m = n = \alpha = 1$, $\text{BKwMO-G}(m,n,a,b,\alpha) \equiv \text{Kw}-\text{G}(a,b)$ (Cordeiro and de Castro, 2011), and (vi) $a = b = \alpha = 1$, $\text{BKwMO-G}(m,n,a,b,\alpha) \equiv \text{B}(m,n)$ (Eugene et al., 2002 and Jones, 2004). In the rest of the article $\text{BKwMO-G}(m,n,a,b,\alpha)$ will by default refer to as BKwMO-G unless specified otherwise.

Genesis of the proposed family of distributions

Here we present a result to show how this new family may arises as a distribution of order statistics of a sample from KwMO-G (a,b,α) distribution.

Theorem1. If *m* and *n* are both integers, then the probability distribution of $\text{BKwMO-G}(m,n,a,b,\alpha)$ arises as distribution of m^{th} order statistics from a random sample size $m+n-1$ from $\text{KwMO-G}(a,b,\alpha)$ distribution.

Proof: Let $T_1, T_2, ..., T_{m+n-1}$ be a random sample of size $m+n-1$ from $\text{KwMO-G}(a,b,\alpha)$ distribution with cdf $1-[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^a\,]^b$. Then the pdf of the m^{th} order statistics $T_{(m)}$ is given by

$$f_{T_{(m)}}(t) = \frac{(m+n-1)!}{(m-1)!\;[(m+n-1)-m]\,!}$$
$$\times\{1-[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^a\,]^b\,\}^{\,m-1}$$
$$\times\{[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^a\,]^b\,\}^{(m+n-1)-m}$$
$$\times[\{\alpha\, a\, b\, g(t) G(t)^{\,a-1}\}/[1-\overline{\alpha}\,\overline{G}(t)]^{a+1}\,]$$
$$\times\{1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^a\,\}^{b-1}$$
$$=[\{a\,b\,\alpha\, g(t)\; G(t)^{\,a-1}\}/\{B(m,n)\,[1-\overline{\alpha}\,\overline{G}(t)]^{\,a+1}\}]$$
$$\times\{1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^a\,\}^{\,bn-1}$$
$$\times\{1-[1-[\,G(t)/\{1-\overline{\alpha}\,\overline{G}(t)\}]^a\,]^b\,\}^{\,m-1}$$

Plots of the pdf and hrf

Here we have plotted the pdf and hrf of the $\text{BKwMO-G}(m,n,a,b,\alpha)$ taking G to be exponential (E), Weibull (W), Lomax (L) and Frechet (Fr) distributions for some chosen values of the parameters to show the variety of shapes assumed by the family.

The BKMO- exponential (BKwMO-E) distribution

When exponential distribution with pdf and cdf $g(t:\lambda) = \lambda e^{-\lambda t}$ and $G(t:\lambda) = 1-e^{-\lambda t}$, $t > 0, \lambda > 0$ is taken as the base line distribution, the pdf and hrf of the resulting BKwMO-E model are respectively given by

$$f^{\text{BKwMOE}}(t;m,n,a,b,\alpha,\lambda)$$
$$= \frac{a\,b\,\alpha\,\lambda e^{-\lambda t}\,[1-e^{-\lambda t}\,]^{a-1}}{B(m,n)\,[1-\overline{\alpha}\,e^{-\lambda t}\,]^{a+1}}\left[1-\left\{\frac{1-e^{-\lambda t}}{1-\overline{\alpha}\,e^{-\lambda t}}\right\}^a\right]^{bn-1}\left[1-\left[1-\left\{\frac{1-e^{-\lambda t}}{1-\overline{\alpha}\,e^{-\lambda t}}\right\}^a\right]^b\right]^{m-1}$$

and $h^{\text{BKwMOE}}(t;m,n,a,b,\alpha,\lambda)$

$$= \frac{a\,b\,\alpha\,\lambda e^{-\lambda t}\,[1-e^{-\lambda t}\,]^{a-1}}{B(m,n)\,[1-\overline{\alpha}\,e^{-\lambda t}\,]^{a+1}[1-I_{1-\left[1-\left\{\frac{1-e^{-\lambda t}}{1-\overline{\alpha}\,e^{-\lambda t}}\right\}^a\right]^b}(m,n)]}[1-\{(1-e^{-\lambda t})/(1-\overline{\alpha}\,e^{-\lambda t})\}^a\,]^{bn-1}$$
$$\times[1-[1-\{(1-e^{-\lambda t})/(1-\overline{\alpha}\,e^{-\lambda t})\}^a\,]^b\,]^{m-1}$$

The BKwMO-Weibull (BKwMO-W) distribution

Considering the Weibull distribution (Weibull, 1951) with parameters $\lambda > 0$ and

$\beta > 0$ having pdf and cdf $g(t) = \lambda\beta\, t^{\beta-1} e^{-\lambda t^{\beta}}$ and $G(t) = 1 - e^{-\lambda t^{\beta}}$ respectively, we get the pdf and hrf of BKwMO- W distribution as

$$f^{\mathrm{BKwMOW}}(t; m, n, a, b, \alpha, \lambda, \beta)$$
$$= \frac{ab\alpha\lambda\beta\, t^{\beta-1} e^{-\lambda t^{\beta}}\, [1-e^{-\lambda t^{\beta}}]^{a-1}}{B(m,n)\,[1-\overline{\alpha}\, e^{-\lambda t^{\beta}}]^{a+1}} \left[1 - \left\{\frac{1-e^{-\lambda t^{\beta}}}{1-\overline{\alpha}\, e^{-\lambda t^{\beta}}}\right\}^{a}\right]^{bn-1} \left[1 - \left[1 - \left\{\frac{1-e^{-\lambda t^{\beta}}}{1-\overline{\alpha}\, e^{-\lambda t^{\beta}}}\right\}^{a}\right]^{b}\right]^{m-1}$$

and $h^{\mathrm{BKwMOW}}(t; m, n, a, b, \alpha, \lambda, \beta)$

$$= \frac{ab\alpha\lambda\beta\, t^{\beta-1} e^{-\lambda t^{\beta}}\, [1-e^{-\lambda t^{\beta}}]^{a-1}}{B(m,n)\,[1-\overline{\alpha}\, e^{-\lambda t^{\beta}}]^{a+1}[1 - I_{1-\left[1-\left\{\frac{1-e^{-\lambda t^{\beta}}}{1-\overline{\alpha}\, e^{-\lambda t^{\beta}}}\right\}^{a}\right]^{b}}(m,n)]} [1 - \{(1-e^{-\lambda t^{\beta}}) / (1-\overline{\alpha}\, e^{-\lambda t^{\beta}})\}^{a}]^{bn-1}$$
$$\times [1 - [1 - \{(1-e^{-\lambda t^{\beta}}) / (1-\overline{\alpha}\, e^{-\lambda t^{\beta}})\}^{a}]^{b}]^{m-1}$$

The BKwMO-Lomax (BKwMO-L) distribution

Considering the Lomax distribution (Lomax, 1954) with pdf and cdf given by $g(t:\beta,\delta) = (\beta/\delta)[1+(t/\delta)]^{-(\beta+1)}$ and $G(t:\beta,\delta) = 1-[1+(t/\delta)]^{-\beta}$, $t > 0$, $\beta > 0$, $\delta > 0$ the pdf and hrf of the BKwMO- L distribution are given respectively by

$$f^{\mathrm{BKwMOL}}(t; m, n, a, b, \alpha, \beta, \delta)$$
$$= \frac{ab\alpha(\beta/\delta)[1+(t/\delta)]^{-(\beta+1)}\, [1-[1+(t/\delta)]^{-\beta}]^{a-1}}{B(m,n)\,[1-\overline{\alpha}[1+(t/\delta)]^{-\beta}]^{a+1}} \left[1 - \left\{\frac{1-[1+(t/\delta)]^{-\beta}}{1-\overline{\alpha}[1+(t/\delta)]^{-\beta}}\right\}^{a}\right]^{bn-1}$$
$$\times \left[1 - \left[1 - \left\{\frac{1-[1+(t/\delta)]^{-\beta}}{1-\overline{\alpha}[1+(t/\delta)]^{-\beta}}\right\}^{a}\right]^{b}\right]^{m-1}$$

and $h^{\mathrm{BKwMOL}}(t; m, n, a, b, \alpha, \beta, \delta)$

$$= \frac{ab\alpha(\beta/\delta)[1+(t/\delta)]^{-(\beta+1)}\, [1-[1+(t/\delta)]^{-\beta}]^{a-1}}{B(m,n)\,[1-\overline{\alpha}[1+(t/\delta)]^{-\beta}]^{a+1}[1 - I_{1-\left[1-\left\{\frac{1-[1+(t/\delta)]^{-\beta}}{1-\overline{\alpha}[1+(t/\delta)]^{-\beta}}\right\}^{a}\right]^{b}}(m,n)]} \left[1 - \left\{\frac{1-[1+(t/\delta)]^{-\beta}}{1-\overline{\alpha}[1+(t/\delta)]^{-\beta}}\right\}^{a}\right]^{bn-1}$$
$$\times \left[1 - \left[1 - \left\{\frac{1-[1+(t/\delta)]^{-\beta}}{1-\overline{\alpha}[1+(t/\delta)]^{-\beta}}\right\}^{a}\right]^{b}\right]^{m-1}$$

The BKwMO-Frechet (BKwMO-Fr) distribution

Suppose the base line distribution is the Frechet distribution (Krishna et al., 2013) with pdf and cdf given by $g(t) = \lambda\delta^{\lambda}\, t^{-(\lambda+1)} e^{-(\delta/t)^{\lambda}}$ and $G(t) = e^{-(\delta/t)^{\lambda}}$, $t > 0$, $\lambda > 0$, $\delta > 0$ respectively, then the corresponding pdf and hrf of BKwMO- Fr distribution becomes

$$f^{\text{BKwMOFr}}(t;m,n,a,b,\alpha,\lambda,\delta)$$
$$=\frac{ab\alpha\lambda\delta^{\lambda}t^{-(\lambda+1)}e^{-(\delta/t)^{\lambda}}[e^{-(\delta/t)^{\lambda}}]^{a-1}}{B(m,n)\,[1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}]^{a+1}}[1-\{e^{-(\delta/t)^{\lambda}}/[1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}]\}^{a}]^{bn-1}$$
$$\times[1-[1-\{e^{-(\delta/t)^{\lambda}}/[1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}]\}^{a}]^{b}]^{m-1}$$

and $h^{\text{BKwMOFr}}(t;m,n,a,b,\alpha,\lambda,\delta)$

$$=\frac{ab\alpha\lambda\delta^{\lambda}t^{-(\lambda+1)}e^{-(\delta/t)^{\lambda}}[e^{-(\delta/t)^{\lambda}}]^{a-1}}{B(m,n)\,[1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}]^{a+1}\left[1-I_{1-\left[1-\left\{\frac{e^{-(\delta/t)^{\lambda}}}{1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}}\right\}^{a}\right]^{b}}(m,n)\right]}$$

$$\times\left[1-\left\{\frac{e^{-(\delta/t)^{\lambda}}}{1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}}\right\}^{a}\right]^{bn-1}\left[1-\left[1-\left\{\frac{e^{-(\delta/t)^{\lambda}}}{1-\bar{\alpha}\{1-e^{-(\delta/t)^{\lambda}}\}}\right\}^{a}\right]^{b}\right]^{m-1}$$

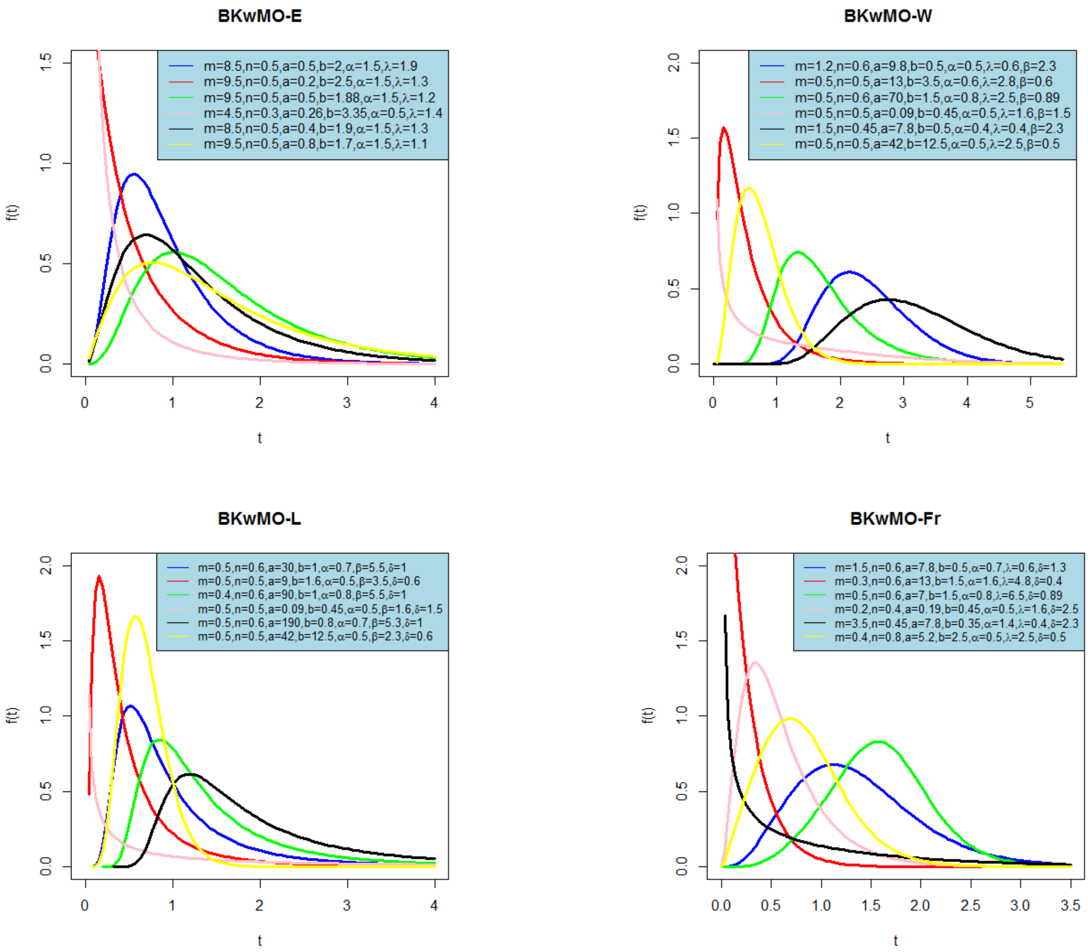


Fig 1 Density plots BKwMO- E, BKwMO- W, BKwMO- L and BKwMO- Fr Distributions clockwise from top left corner.

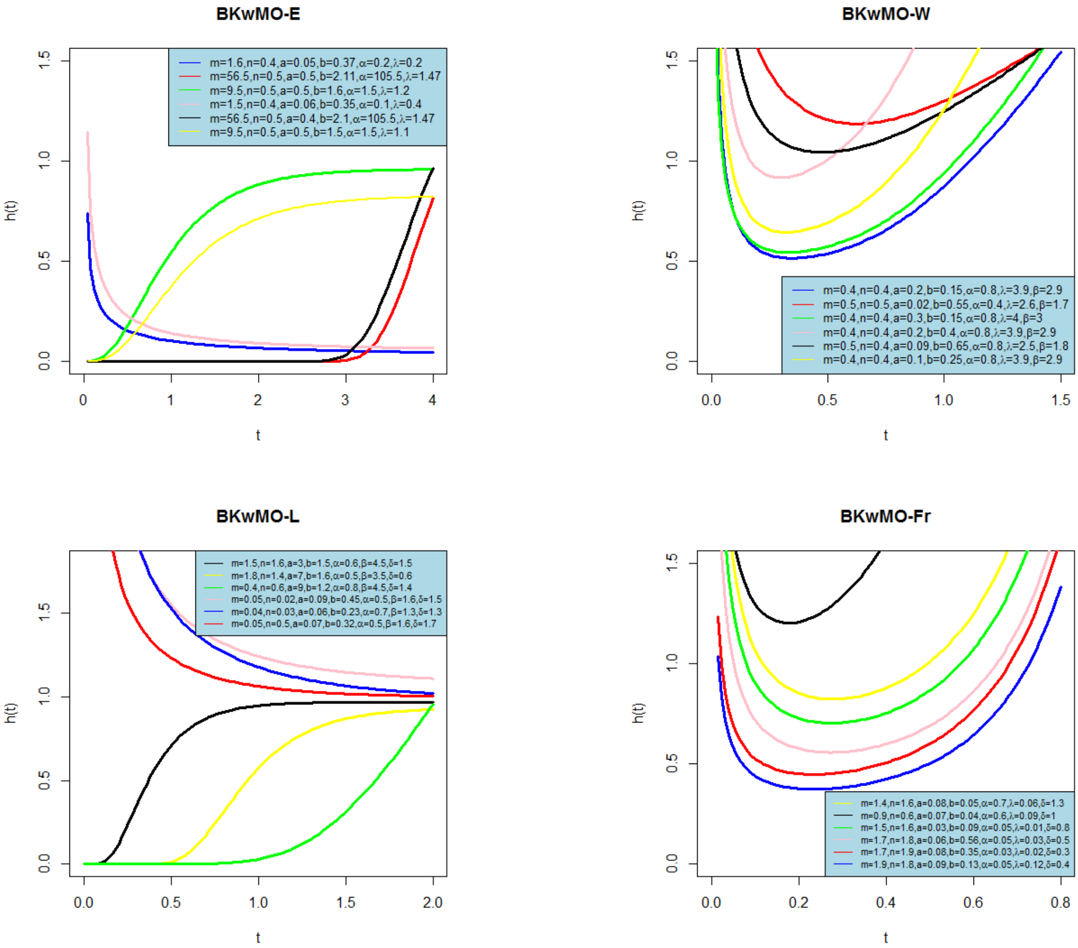


Fig 2 Hazard plots BKwMO- E, BKwMO- W, BKwMO- L and BKwMO- Fr distributions clockwise from top left corner.

From the plots in figure 1 and 2 it can be seen that the family is very flexible and can offer many different types of shapes of density and hazard rate function including the bathtub shape for hazard.

Statistical and mathematical properties

In this section we derive some general results for the proposed BKwMO- G(m,n,a,b,α) family.

Series expansions of pdf and cdf

By using binomial expansion in (4), we obtain

$$f^{\text{BKwMOG}}(t;m,n,a,b,\alpha)$$

$$= f^{\text{MO}}(t;\alpha)\sum_{i=0}^{m-1}\sum_{j=0}^{b(i+n)-1}\beta_{i,j} \times [F^{\text{MO}}(t;\alpha)]^{a(j+1)-1} \quad (7)$$

$$= \sum_{i=0}^{m-1}\sum_{j=0}^{b(i+n)-1}\beta'_{i,j}\frac{d}{dt}[F^{\text{MO}}(t;\alpha)]^{a(j+1)} \quad (8)$$

where $\beta'_{i,j} = \dfrac{b}{B(m,n)(j+1)}\dbinom{m-1}{i} \times \dbinom{b(i+n)-1}{j}(-1)^{i+j}$,

$\beta_{i,j} = a(j+1)\beta'_{i,j}$ and $f^{\text{MO}}(t;\alpha)$, $F^{\text{MO}}(t;\alpha)$ are pdf and cdf respectively of the Marshall-Olkin (MO) family.

Alternatively, we can expand the pdf as

$$f^{\text{BKwMOG}}(t;m,n,a,b,\alpha)$$

$$= f^{MO}(t;\alpha)\sum_{k=0}^{a(j+1)-1}\gamma_k[\bar{F}^{\text{MO}}(t;\alpha)]^k \quad (9)$$

$$= \sum_{k=0}^{a(j+1)-1}\gamma'_k f^{\text{MO}}(t;\alpha(k+1))$$

where, $\gamma'_k = \sum_{i=0}^{m-1}\sum_{j=0}^{b(i+n)-1}(-1)^k\beta_{i,j}$

$\times\binom{a(j+1)-1}{k}$ and $\gamma_k = (k+1)\,\gamma'_k$

Similarly an expansion for the cdf of BKwMO-G(m,n,a,b,α) can be derived as

$$F^{\text{BKwMOG}}(t;m,n,a,b,\alpha)$$

$$= I_{1-[1-\{G(t)/1-\bar{\alpha}\bar{G}(t)\}^a]^b}(m,n)$$

$$= I_{1-[1-F^{\text{MO}}(t;\alpha)^a]^b}(m,n)$$

$$= \sum_{i=m}^{m+n-1}\binom{m+n-1}{i}\left\{1-[1-F^{\text{MO}}(t;\alpha)^a]^b\right\}^i$$

$$\times\left\{[1-F^{\text{MO}}(t;\alpha)^a]^b\right\}^{m+n-1-i}$$

$$= \sum_{i=m}^{m+n-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{ak}\eta_{i,j,k,l}\,\bar{F}^{\text{MO}}(t;\alpha)^l$$

where,

$$\eta_{i,j,k,l} = (-1)^{j+k+l}\binom{m+n-1}{i}\binom{i}{j}$$

$$\times\binom{b(m+n-i+j-1)}{k}\binom{ak}{l}$$

Now to expand the cdf $F^{BKwMOG}(t;m,n,a,b,\alpha)$ $= I_{1-[1-[G(t)/\{1-\bar{\alpha}\bar{G}(t)\}]^a]^b}(m,n)$, we use the following result

$$I_z(a,b) = \frac{B_z(a,b)}{B(a,b)}$$

$$= \frac{z^a}{B(a,b)}\sum_{i=0}^{\infty}\binom{b-1}{i}\frac{(-1)^i}{(a+i)}z^i \quad (10)$$

(See “Incomplete Beta Function” *From Math World*-A Wolfram Web Resource. http://mathworld.Wolfram.com/Incomplete Beta Function. html).

From (5) using (10) we get

$$F^{\text{BKwMOG}}(t;m,n,a,b,\alpha)$$

$$= \sum_{i,j,k,p=0}^{\infty}\sum_{l=p}^{\infty}\frac{(-1)^{i+j+k+l+p}}{B(m,n)(m+i)}\binom{n-1}{i}\binom{m+i}{j}$$

$$\times\binom{bj}{k}\binom{ak}{l}\binom{l}{p}[F^{\text{MO}}(t;\alpha)]^p$$

$$= \sum_{p=0}^{\infty}\lambda_p\,[F^{\text{MO}}(t;\alpha)]^p \quad (11)$$

where, $\lambda_p = \sum_{i,j,k=0}^{\infty}\sum_{l=p}^{\infty}\frac{(-1)^{i+j+k+l+p}}{B(m,n)(m+i)}\binom{n-1}{i}$

$$\times\binom{m+i}{j}\binom{bj}{k}\binom{ak}{l}\binom{l}{p}$$

Order statistics

Suppose $T_1, T_2, ..., T_\vartheta$ is a random sample from any BKwMO-G(m,n,a,b,α) distribution. Let $T_{r:\vartheta}$ denote the r^{th} order statistics. The pdf of $T_{r:\vartheta}$ can be expressed as

$$f_{r:\vartheta}(t) = \frac{\vartheta!}{(r-1)!\,(\vartheta-r)!}\sum_{j=0}^{\vartheta-r}(-1)^j\binom{\vartheta-r}{j}$$

$$\times f^{\text{BKwMO}}(t)F^{\text{BKwMO}}(t)^{j+r-1}$$

We can now use general expansion of the pdf and cdf in equation last section to get

$$f_{r:n}(t) = \frac{\vartheta!\,f^{MO}(t;\alpha)}{(r-1)!\,(\vartheta-r)!}\sum_{j=0}^{\vartheta-r}(-1)^j\binom{\vartheta-r}{j}$$

$$\times\left\{\sum_{i=0}^{m-1}\sum_{q=0}^{b(i+n)-1}\beta_{i,q}\,[F^{MO}(t;\alpha)]^{a(q+1)-1}\right\}$$

$$\times\left\{\sum_{p=0}^{\infty}d_{j+r-1,p}[F^{MO}(t;\alpha)]^p\right\}$$

$$= \frac{\vartheta!\,f^{MO}(t;\alpha)}{(r-1)!\,(\vartheta-r)!}\sum_{j=0}^{\vartheta-r}(-1)^j\binom{\vartheta-r}{j}\sum_{i=0}^{m-1}\sum_{p=0}^{\infty}$$

$$\left\{ \sum_{q=0}^{b(i+n)-1} \beta_{i,q}\, d_{j+r-1,p}\, [F^{MO}(t;\alpha)]^{a(q+1)+p-1} \right\}$$

$$= f^{\text{MO}}(t;\alpha) \sum_{i=0}^{m-1} \sum_{p=0}^{\infty}$$

$$\times \sum_{q=0}^{b(i+n)-1} \xi_{i,p,q} [F^{\text{MO}}(t;\alpha)]^{a(q+1)+p-1} \quad (12)$$

where, $\xi_{i,p,q} = \dfrac{\vartheta!}{(r-1)!\,(\vartheta-r)!} \sum_{j=0}^{\vartheta-r} (-1)^j \times \binom{\vartheta-r}{j} \beta_{i,q}\, d_{j+r-1,p}$

wherein

$$d_{j+r-1\,p} = \frac{1}{p\lambda_0} \sum_{c=1}^{p} [c(j+r)-p]\lambda_c d_{j+r-1,p-c}$$

(Nadarajah et al., 2015) and $\beta_{i,q}$, λ_c defined earlier.

Probability weighted moments

In this section we express probability weighted moments (PWM) of the proposed family in terms of those of $\text{MO}(\alpha)$ distribution.

The $(p,q,r)^{th}$ PWM of T is defined by

$$\Gamma_{p,q,r} = \int_{-\infty}^{\infty} t^p F(t)^q\, [1-F(t)]^r\, f(t)\, dt$$

(Greenwood *et al.,* 1979). From equations (7) and (9) the s^{th} moment of T can be obtained either as

$$E(T^s) = \int_0^{\infty} t^s\, f^{\text{BKwMOG}}(t;m,n,a,b,\alpha)\, dt$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{b(i+n)-1} \beta_{i,j}\, \Gamma_{s,a(j+1)-1,0}$$

and $E(T^s) = \displaystyle\sum_{k=0}^{a(j+1)-1} \gamma_k \int_0^{\infty} t^s\, [\bar{F}^{\text{MO}}(t;\alpha)]^k \times f^{\text{MO}}(t;\alpha)\, dt$

$$= \sum_{k=0}^{a(j+1)-1} \gamma_k\, \Gamma_{s,0,k}$$

where $\Gamma_{p,q,r}$

$$= \int_0^{\infty} t^p\, F^{\text{MO}}(t;\alpha)^q\, \bar{F}^{\text{MO}}(t;\alpha)^r\, f^{\text{MO}}(t;\alpha)\, dt$$

is the PWM of $\text{MO}(\alpha)$ distribution.

Proceeding similarly we can derive s^{th} moment of the r^{th} order statistic $T_{r:\vartheta}$ from a random sample of size ϑ from $\text{BKwMO-G}(m,n,a,b,\alpha)$ using equation (12) as

$$E(T^s{}_{r;\vartheta}) = \sum_{i=0}^{m-1} \sum_{p=0}^{\infty} \sum_{q=0}^{b(i+n)-1} \xi_{i,p,q}\, \Gamma_{s,\,a(q+1)+p-1,\,0},$$

where $\beta_{i,j}$, γ_k and $\xi_{i,p,q}$ defined earlier.

Generating function

Here we express the moment generating function of the proposed family in terms of those of the exponentiated $\text{MO}(\alpha)$ distribution using the results of section 3.1 as

$$M_T(s) = \text{E}[e^{sT}]$$

$$= \int_0^{\infty} e^{st} f^{\text{BKwMOG}}(t;m,n,a,b,\alpha)\, dt$$

$$= \int_0^{\infty} e^{st} \sum_{i=0}^{m-1} \sum_{j=0}^{b(i+n)-1} \beta'_{i,j}\, \frac{d}{dt} [F^{\text{MO}}(t;\alpha)]^{a(j+1)}\, dt$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{b(i+n)-1} \beta'_{i,j} \int_0^{\infty} e^{st} \frac{d}{dt} [F^{\text{MO}}(t;\alpha)]^{a(j+1)}\, dt$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{b(i+n)-1} \beta_{i,j}\, M_X(s),$$

where $M_X(s)$ is the mgf of an exponentiated (Lehman Alternative-I) $\text{MO}(\alpha)$ distribution.

Rényi entropy

The entropy of a random variable is a measure of uncertainty variation and has been used in various situations in science and engineering. The Rényi entropy of a random variable having pdf $f(t)$ is given by

$$I_R(\tau) = (1-\tau)^{-1}\log\left(\int_{-\infty}^{\infty} f(t)^{\tau}\, dt\right)$$

where $\tau > 0$ and $\tau \neq 1$. For further details, see Song (2001). Here

$$I_R(\tau) = (1-\tau)^{-1}\log\left(\sum_{i=0}^{\tau(m-1)}\ \sum_{j=0}^{b(i+\tau n)-\tau}\eta_{i,j}\int_0^{\infty} f^{\,\mathrm{MO}}(t;\alpha)^{\tau}\,[F^{\,\mathrm{MO}}(t;\alpha)]^{a(j+\tau)-\tau}\, dt\right),$$

where $\eta_{i,j} = \dfrac{(ab)^{\tau}}{B(m,n)^{\tau}}\binom{\tau(m-1)}{i}\binom{b(i+\tau n)-\tau}{j}(-1)^{i+j}$

Quantile function, Median and random sample generation

The quantile function can be obtained by inverting cdf in equation (5)
$t = Q(u)$

$$= Q_G\left(\frac{(1-\alpha)[1-(1-Q_{m,n}(u))^{1/b}]^{1/a}}{1-\alpha\,[1-[1-(1-Q_{m,n}(u))^{1/b}]^{1/a}]}\right)$$

where $z = Q_{m,n}(u)$ is the quantile function of beta distribution.

It is possible to obtain some expansions for $Q_{m,n}(u)$ in the Wolfram website **(**http:// mathworld. wolfram.com/ PowerSeries.html) as $z = Q_{m,n}(u) = \sum_{i-0}^{\infty} e_i\, u^{\,i/m}$, where

$e_i = [m\,B(m,n)]^{1/m}\, d_i$,

$d_0 = 0,\ d_1 = 1,\ d_2 = (n-1)/(m+1)$,

$d_3 = \{(n-1)(m^2 + 3mn - m + 5n - 4)\}/$
$\{2(m+1)^2(m+2)\}$

$d_4 = (n-1)[m^4 + (6n-1)m^3$
$+ (n+2)(8n-5)m^2 + (33n^2 - 30n + 4)m$
$+ n(31n-47) + 18]/$
$[3(m+1)^3(m+2)(m+3)]$, etc.

For example, the p^{th} quantile t_p and median of BKwMO- E are respectively given by

$t_p =$

$$-\frac{1}{\lambda}\log\left[1-\frac{(1-\alpha)[1-(1-Q_{m,n}(p))^{\frac{1}{b}}]^{\frac{1}{a}}}{1-\alpha\,[1-[1-(1-Q_{m,n}(p))^{\frac{1}{b}}]^{\frac{1}{a}}]}\right]$$

and $t_{med} =$

$$-\frac{1}{\lambda}\log\left[1-\frac{(1-\alpha)[1-(1-Q_{m,n}(0.5))^{\frac{1}{b}}]^{\frac{1}{a}}}{1-\alpha\,[1-[1-(1-Q_{m,n}(0.5))^{\frac{1}{b}}]^{\frac{1}{a}}]}\right].$$

Random sample from BKwMO– G can be generated by inverting the cdf.

Skewness and kurtosis

The Bowley's skewness (Kenney and Keeping, 1962) measures and Moor's kurtosis (Moors, 1988) measure for BKwMO– G family are respectively given in terms of the quantile function as

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)} \quad \text{and}$$

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}$$

Asymptotes

Some results regarding the asymptotic shapes of the proposed family following is stated here.

Proposition 1 The asymptotes of pdf, cdf and hrf given in equations (4), (5) and (6) as $t \to 0$ are given by

$$f(t) \sim [\{ab\ g(t)\ G(t)^{a-1}\}/\{B(m,n)\ \alpha^{a}\}]\,[1-\{1-\{G(t)/\alpha\}^{a}\}^{b}]^{m-1} \quad \text{as } t \to 0$$

$$F(t) \sim (1/B(m,n)m)[1-\{1-\{G(t)/\alpha\}^{a}\}^{b}]^{m} \quad \text{as } t \to 0$$

$$h(t) \sim [\{ab\ g(t)\ G(t)^{a-1}\}/\{B(m,n)\ \alpha^{a}\}]\,[1-\{1-\{G(t)/\alpha\}^{a}\}^{b}]^{m-1} \quad \text{as } t \to 0$$

Proposition 2 The asymptotes of pdf, sf and hrf given in equations (4), (5) and (6) as $t \to \infty$ are given by

$$f(t) \sim \{(\alpha a b g(t))/B(m,n)\}[1-[1/\{1-\bar{\alpha}\bar{G}(t)\}]^{a}]^{bn-1} \quad \text{as } t \to \infty$$

$$1-F(t) \sim (1/nB(m,n))[1-[1/\{1-\bar{\alpha}\bar{G}(t)\}]^{a}]^{bn} \quad \text{as } t \to \infty$$

$$h(t) \sim \alpha a b n g(t)[1-[1/\{1-\bar{\alpha}\bar{G}(t)\}]^{a}]^{-1} \quad \text{as } t \to \infty$$

Maximum likelihood estimation and data modelling

The model parameters of the BKwMO-G(m,n,a,b,α) distribution can be estimated by maximum likelihood. Let $t=(t_1,t_2,...,t_{\vartheta})$ be a random sample of size ϑ from BKwMO-G(m,n,a,b,α) with parameter vector $\boldsymbol{\rho}=(m,n,a,b,\alpha,\boldsymbol{\beta})^{T}$, where $\boldsymbol{\beta}=(\beta_1,\beta_2,...,\beta_q)$ corresponds to the parameter vector of the baseline distribution G. Then the log-likelihood function for $\boldsymbol{\rho}$ is given by

$$\ell=\ell(\boldsymbol{\rho})=\vartheta\log(ab\alpha)+\sum_{i=1}^{\vartheta}\log[g(t_i,\boldsymbol{\beta})]+(a-1)\sum_{i=1}^{\vartheta}\log[G(t_i,\boldsymbol{\beta})]-\vartheta\log[B(m,n)]$$
$$-(a+1)\sum_{i=1}^{\vartheta}\log[1-\bar{\alpha}\bar{G}(t_i,\boldsymbol{\beta})]+(bn-1)\sum_{i=1}^{\vartheta}\log[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\bar{G}(t_i,\boldsymbol{\beta})\}]^{a}]$$
$$+(m-1)\sum_{i=1}^{\vartheta}\log[1-[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\bar{G}(t_i,\boldsymbol{\beta})\}]^{a}]^{b}] \quad (13)$$

This log-likelihood function can not be solved analytically because of its complex form here we have employed global numerical optimization methods to obtain the maximum likelihood estimates (MLEs). By taking the partial derivatives of the log-likelihood function with respect to m,n,a,b,α and $\boldsymbol{\beta}$ components of the score vector $U_{\boldsymbol{\rho}}=(U_m,U_n,U_a,U_b,U_{\alpha},U_{\boldsymbol{\beta}^{T}})^{T}$ can be obtained as follows:

$$U_m=\frac{\partial\ell}{\partial m}=-\vartheta\psi(m)+\vartheta\psi(m+n)+\sum_{i=1}^{\vartheta}\log[1-[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\bar{G}(t_i,\boldsymbol{\beta})\}]^{a}]^{b}]$$

$$U_n=\frac{\partial\ell}{\partial n}=-\vartheta\psi(n)+\vartheta\psi(m+n)+b\sum_{i=1}^{\vartheta}\log[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\bar{G}(t_i,\boldsymbol{\beta})\}]^{a}]$$

$$U_a=\frac{\partial\ell}{\partial a}=\frac{\vartheta}{a}+\sum_{i=1}^{\vartheta}\log[G(t_i,\boldsymbol{\beta})]-\sum_{i=1}^{\vartheta}\log[1-\bar{\alpha}\bar{G}(t_i,\boldsymbol{\beta})]$$

$$+(1-bn)\sum_{i=1}^{\vartheta}\frac{[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a\log[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]}{1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a}$$

$$+b(m-1)\sum_{i=1}^{\vartheta}\frac{[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a]^{b-1}[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a}{1-[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a]^b}$$

$$\times\log[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]$$

$$U_b=\frac{\partial\ell}{\partial b}=\frac{\vartheta}{b}+n\sum_{i=1}^{\vartheta}\log[\,1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a\,]$$

$$+(1-m)\sum_{i=1}^{\vartheta}\frac{[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a]^b\log[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a]}{1-[1-[G(t_i,\boldsymbol{\beta})/\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}]^a]^b}$$

$$U_\alpha=\frac{\partial\ell}{\partial\alpha}=\frac{\vartheta}{\alpha}-(a+1)\sum_{i=1}^{\vartheta}\frac{\overline{G}(t_i,\boldsymbol{\beta})}{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})}$$

$$+a(bn-1)\sum_{i=1}^{\vartheta}\frac{G(t_i,\boldsymbol{\beta})^a\,\overline{G}(t_i,\boldsymbol{\beta})}{[\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^a-G(t_i,\boldsymbol{\beta})^a][1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})]}$$

$$+ab(m-1)\sum_{i=1}^{\vartheta}\frac{[\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^a-G(t_i,\boldsymbol{\beta})^a]^b G(t_i,\boldsymbol{\beta})^a\,\overline{G}(t_i,\beta)}{[\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^{ab}-\{\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^a-G(t_i,\boldsymbol{\beta})^a\}^b][1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})]}$$

$$U_{\boldsymbol{\beta}}=\frac{\partial\ell}{\partial\beta}=\sum_{i=1}^{\vartheta}\frac{g^{(\beta)}(t_i,\boldsymbol{\beta})}{g(t_i,\boldsymbol{\beta})}+(a-1)\sum_{i=1}^{\vartheta}\frac{G^{(\beta)}(t_i,\boldsymbol{\beta})}{G(t_i,\boldsymbol{\beta})}-(a+1)\sum_{i=1}^{\vartheta}\frac{\bar{\alpha}\,G^{(\beta)}(t_i,\boldsymbol{\beta})}{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})}$$

$$+a\alpha(1-bn)\sum_{i=1}^{\vartheta}\frac{G(t_i,\boldsymbol{\beta})^{a-1}G^{(\beta)}(t_i,\boldsymbol{\beta})}{[\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^a-G(t_i,\boldsymbol{\beta})^a][1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})]}$$

$$+ab\alpha(m-1)\sum_{i=1}^{\vartheta}\frac{[\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^a-G(t_i,\boldsymbol{\beta})^a]^b G(t_i,\boldsymbol{\beta})^{a-1}G^{(\beta)}(t_i,\boldsymbol{\beta})}{[\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^{ab}-\{\{1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})\}^a-G(t_i,\boldsymbol{\beta})^a\}^b][1-\bar{\alpha}\,\overline{G}(t_i,\boldsymbol{\beta})]}$$

Asymptotic standard error for the MLEs

The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\boldsymbol{\rho})$ which can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The $i\,j^{th}$ elements of $\mathrm{I}_\vartheta(\boldsymbol{\rho})$ are given by

$$\mathrm{I}_{i\,j}=-E[\partial^2\,\ell(\boldsymbol{\rho})/\partial\rho_i\,\partial\rho_j]\ ,$$

$$i,\ j=1,\ 2,\ \cdots,\ 5+q$$

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_\vartheta(\boldsymbol{\rho})$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_\vartheta(\hat{\boldsymbol{\rho}})=(\hat{\mathrm{I}}_{i\,j})$ is defined as:

$$\hat{\mathrm{I}}_{i\,j}\approx\left(-\partial^2\,\ell(\boldsymbol{\rho})/\partial\rho_i\,\partial\rho_j\right)_{\rho=\hat{\rho}}\ ,$$

$$i,\ j=1,\ 2,\ \cdots,\ 5+q$$

Using the general theory of MLEs under some regularity conditions on the parameters as $\vartheta\to\infty$ the asymptotic distribution of $\sqrt{\vartheta}\,(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})$ is $N_k(0,V_\vartheta)$, where

$$V_\vartheta=(v_{jj})=\mathrm{I}_\vartheta^{-1}(\boldsymbol{\rho})\,.$$

The asymptotic behaviour remains valid if V_ϑ is replaced by $\hat{V}_\vartheta=\hat{\mathrm{I}}^{-1}(\hat{\boldsymbol{\rho}})$. This result can be used to provide large sample standard errors for the model parameters. Thus an approximate standard error for the MLE of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{jj}}$.

Comparative data modelling

Comparison with the KwMO-G and BMO-G

Here we consider fitting of three real data sets to show that the BKwMO-W distribution from the BKwMO-W family can be a better model than its sub model the KwMO-W and BMO-W (Alizadeh et al., 2015 a, b) distribution.

The density function of the KwMO-W and BMO-W are respectively given by

$$f^{\text{KwMOW}}(t;a,b,\alpha,\lambda,\beta) = \frac{ab\,\alpha\lambda\beta\, t^{\beta-1} e^{-\lambda t^\beta}[1-e^{-\lambda t^\beta}]^{a-1}}{[1-\overline{\alpha}\,e^{-\lambda t^\beta}]^{a+1}} \times \left[1-\left\{\frac{1-e^{-\lambda t^\beta}}{1-\overline{\alpha}\,e^{-\lambda t^\beta}}\right\}^a\right]^{b-1}$$

$-\infty < t < \infty;\ \alpha > 0; a > 0, b > 0, \lambda > 0, \beta > 0$

and $f^{\text{BMOW}}(t;m,n,\alpha,\lambda,\beta) =$

$$\frac{\alpha^n \lambda\beta\, t^{\beta-1} e^{-\lambda t^\beta}[1-e^{-\lambda t^\beta}]^{m-1}[e^{-\lambda t^\beta}]^{n-1}}{B(m,n)[1-\overline{\alpha}\,e^{-\lambda t^\beta}]^{m+n}}$$

$-\infty < t < \infty;\ \alpha > 0; m > 0, n > 0, \lambda > 0, \beta > 0$

We have used AIC, BIC, CAIC and HQIC for model selection, KS test for goodness of fit and Likelihood ratio test for test of hypotheses for nested models.

Likelihood Ratio Test for nested models

We have seen that the BKwMO–W$(m,n,a,b,\alpha;\lambda,\beta)$ reduces to KwMO–W$(a,b,\alpha;\lambda,\beta)$, when $m = n = 1$ and to BMO-W$(m,n,\alpha;\lambda,\beta)$ for $a = b = 1$, we have therefore used likelihood ratio test to check whether the additional parameter(s) in the proposed model provide statistically significant improvement in data fitting over these nested in sub models.

Here we have employed likelihood ratio criterion to test the following null hypothesis:

(i) $H_0: m = n = 1$, that is the sample is from KwMO–W$(a,b,\alpha,\lambda,\beta)$

$H_1: m \neq 1, n \neq 1$, that is the sample is BKwMO–W$(m,n,a,b,\alpha,\lambda,\beta)$.

(ii) $H_0: a = b = 1$, that is the sample is from BMO-W$(m,n,\alpha,\lambda,\beta)$

$H_1: a \neq 1, b \neq 1$, that is the sample is BKwMO–W$(m,n,a,b,\alpha,\lambda,\beta)$.

Writing $\boldsymbol{\rho} = (m,n,a,b,\alpha,\lambda,\beta)$ the likelihood ratio test statistic is given by LR = $-2\ln(\ell(\hat{\boldsymbol{\rho}}^*)/\ell(\hat{\boldsymbol{\rho}}))$, where $\hat{\boldsymbol{\rho}}^*$ is the restricted ML estimates under the null hypothesis H_0 and $\hat{\boldsymbol{\rho}}$ is the unrestricted ML estimates under the alternative hypothesis H_1. Here under the null hypothesis H_0 the LR criterion follows Chi-square distribution with 2 (two) degrees of freedom (df). The null hypothesis is rejected for *p*-value less than 0.05.

Data set I: This data set is obtained from Smith and Naynlor (1987). The data consists of 63 observations of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England.

Data set II: This data set is a subset of data reported by Bekker et al., (2000) which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consisting of survival times (in years) for 46 patients.

Data set III: This data set about 346 nicotine measurements made from several brands of cigarettes in 1998. The data have been collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection**.** [http: //www.ftc.gov/ reports/tobacco or http: // pw1.netcom.com/ rdavis2/ smoke. html.]

The total time on test (TTT) plot proposed by Aarset (1987) is technique helpful in detecting the shape of hazard rate of observed data and hence in deciding a model for fitting. The TTT is drawn by plotting

$$T(i/\vartheta)=\left\{\sum_{r=1}^{i} y_{(r)}+(\vartheta-i)y_{(i)}\right\}\Big/\sum_{r=1}^{\vartheta} y_{(r)}$$

against i/ϑ, where, $i=1,2,...,\vartheta$ and $y_{(r)}\ (r=1,2,...,\vartheta)$ are the order statistics corresponding to the sample, The hazard of the given data set is constant, decreasing and increasing if the shape of the TTT plot is a straight diagonal line, is of convex shape and concave shape respectively. The TTT plots for the data sets considered here are presented in Fig. 3 which indicates that the data set I and III have increasing hazard rate while for data set II it is nearly constant. We have presented the descriptive statistics of the data sets in Table 1 and findings of the data fittings in Table 2(a) and 2(b).

In Table 2(b), the MLEs, log likelihood, AIC, BIC, CAIC and HQIC values and KS, LR statistics are presented for both distributions. According to the lowest values of the AIC, BIC, CAIC and HQIC the BKwMO-W distribution is seen as the better model than both the KwMO-W and BMO-W distribution. Histograms and ogives of the data sets I, II and III along with the fitted pdfs and cdf's are also displayed in Figures 4, 5 and 6 respectively to visually confirm the closeness of the fitted distributions to the observed data. It's easy to see that BKwMO-W provides the best fit to all the data sets considered here.

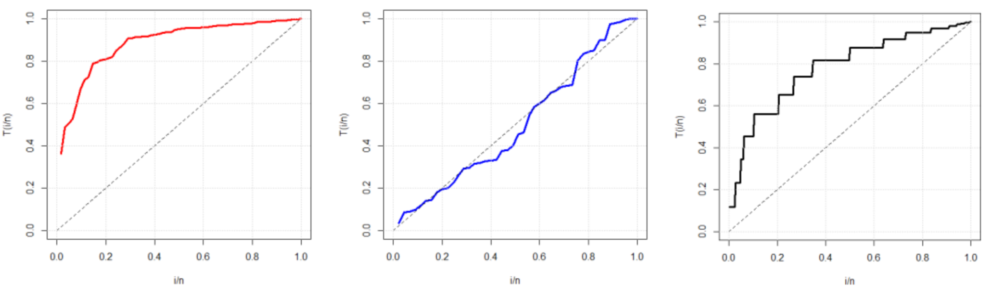


Fig 3: TTT plots of the data sets I, II and III form left to right

Table 1: Descriptive Statistics for data set I, II and III

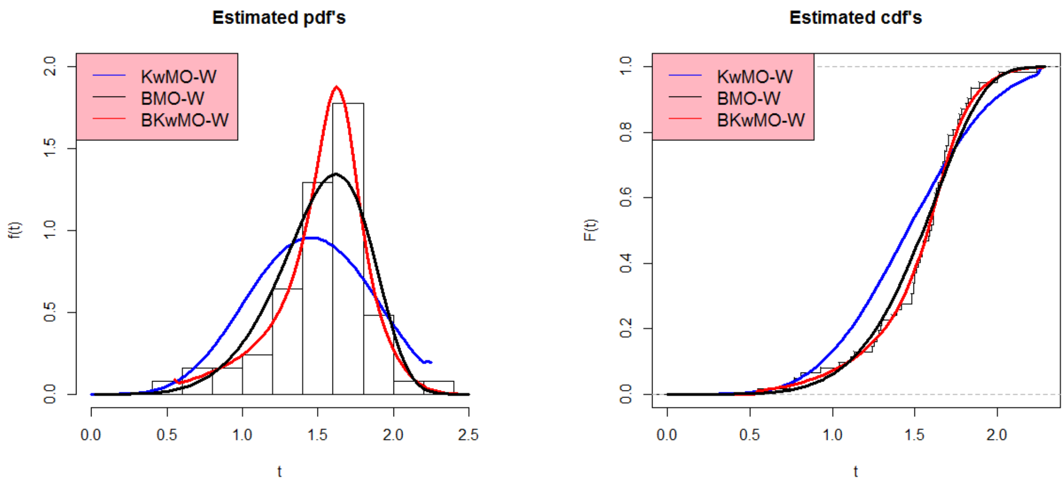
Data Sets	Minimum	Mean	Median	s.d.	Skewness	Kurtosis	1st Qu.	3rd Qu.	Maximum
I	0.550	1.507	1.590	0.324	-0.879	0.800	1.375	1.685	2.240
II	0.047	1.343	0.841	1.246	0.936	-0.457	0.395	2.178	4.033
III	0.100	0.853	0.900	0.334	0.171	0.296	0.600	1.100	2.000

Table 2 (a): MLEs, standard errors (in parentheses) values for data sets I, II and III

Models	$\hat{m}$	$\hat{n}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$
Data I							
BMO-W	0.941	14.606	---	---	3.512	0.014	5.419
$(m,n,\alpha,\lambda,\beta)$	(0.302)	(3.659)			(1.361)	(0.006)	(0.233)
KwMO-W	---	---	1.367	0.103	2.144	1.738	3.768
$(a,b,\alpha,\lambda,\beta)$			(0.591)	(0.014)	(0.629)	(0.002)	(0.001)
BKwMO-W	0.125	0.884	4.012	0.249	12.818	0.149	6.283
$(m,n,a,b,\alpha,\lambda,\beta)$	(0.016)	(0.229)	(1.245)	(0.123)	(5.786)	(0.002)	(2.347)
Data II							
BMO-W	1.733	0.223	---	---	0.016	0.643	1.921
$(m,n,\alpha,\lambda,\beta)$	(0.325)	(0.089)			(0.025)	(0.134)	(0.513)
KwMO-W	---	---	0.326	0.149	0.0006	0.059	3.931
$(a,b,\alpha,\lambda,\beta)$			(0.142)	(0.032)	(0.001)	(1.581)	(2.367)
BKwMO-W	0.093	0.081	2.288	1.499	0.0004	0.003	5.931
$(m,n,a,b,\alpha,\lambda,\beta)$	(0.036)	(0.016)	(1.183)	(1.068)	(0.003)	(0.010)	(4.783)
Data III							
BMO-W	0.237	0.917	---	---	0.014	0.005	8.571
$(m,n,\alpha,\lambda,\beta)$	(0.021)	(0.181)			(0.006)	(0.001)	(0.377)
KwMO-W	---	---	0.438	0.883	0.124	0.110	5.470
$(a,b,\alpha,\lambda,\beta)$			(0.096)	(0.198)	(0.089)	(0.097)	(0.883)
BKwMO-W	0.421	0.308	0.443	1.719	0.002	0.003	10.952
$(m,n,a,b,\alpha,\lambda,\beta)$	(0.537)	(0.365)	(0.541)	(2.380)	(0.015)	(0.0009)	(0.014)

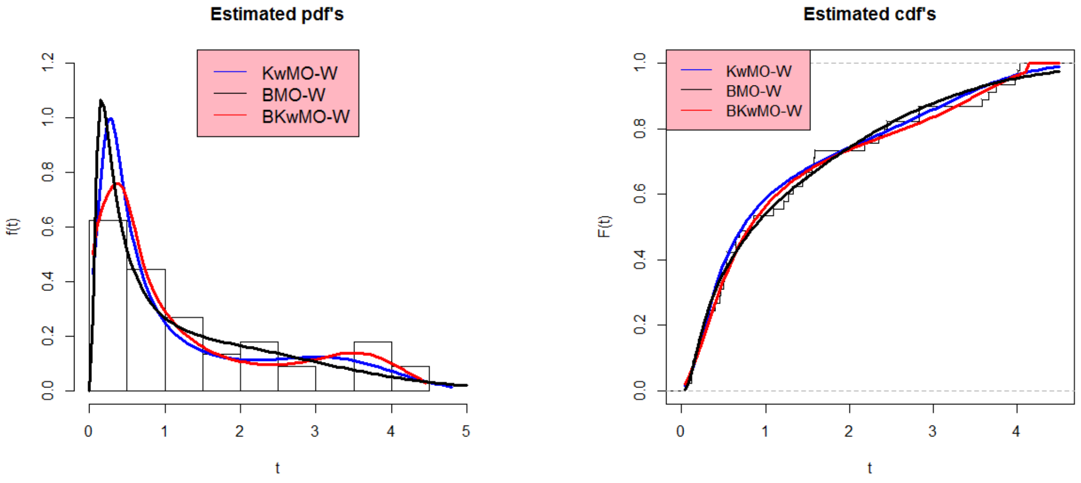
Table 2 (b): AIC, BIC, CAIC, HQIC, KS (*p*-value) and LR (*p*-value) values for data sets I, II and III

Models	$-l_{max}$	AIC	BIC	CAIC	HQIC	KS (*p*-value)	LR (*p*-value)
Data I							
BMO-W $(m,n,\alpha,\lambda,\beta)$	16.43	42.86	53.56	43.91	47.06	0.08 (0.91)	17.66 (0.0001)
KwMO-W $(a,b,\alpha,\lambda,\beta)$	19.68	49.36	60.06	50.41	53.56	0.12 (0.42)	24.16 (0.014)
BKwMO-W $(m,n,a,b,\alpha,\lambda,\beta)$	7.60	**29.20**	**44.18**	**31.23**	**35.08**	0.07 (0.95)	---
Data II							
BMO-W $(m,n,\alpha,\lambda,\beta)$	56.43	122.86	132.01	124.36	126.26	0.08 (0.91)	8.84 (0.012)
KwMO-W $(a,b,\alpha,\lambda,\beta)$	55.97	121.94	131.09	123.44	125.34	0.58 (0.82)	7.92 (0.019)
BKwMO-W $(m,n,a,b,\alpha,\lambda,\beta)$	52.01	**118.02**	**130.83**	**120.97**	**122.78**	0.54 (0.92)	---
Data IIII							
BMO-W $(m,n,\alpha,\lambda,\beta)$	109.54	229.08	248.33	229.26	236.78	0.30 (0.06)	7.26 (0.026)
KwMO-W $(a,b,\alpha,\lambda,\beta)$	111.49	232.98	252.23	233.15	240.68	0.26 (0.12)	11.16 (0.004)
BKwMO-W $(m,n,a,b,\alpha,\lambda,\beta)$	105.91	**225.82**	**252.77**	**226.15**	**236.60**	0.25 (0.14)	---



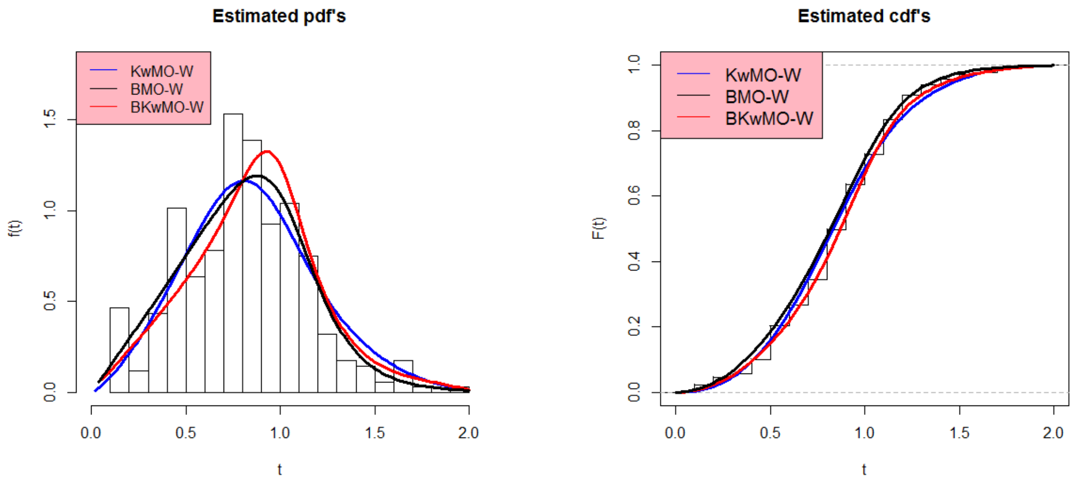
(a) (b)

Fig 4: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the KwMO−W, BMO−W and BKwMO−W for data set I.



(a) (b)

Fig 5: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the $KwMO-W$, $BMO-W$ and $BKwMO-W$ for data set II.



(a) (b)

Fig 6: Plots of the (a) observed histogram and estimated pdf's and (b) observed ogive and estimated cdf's for the $KwMO-W$, $BMO-W$ and $BKwMO-W$ for data set III.

From the findings tabulated in Table 2(b), on the basis of AIC, BIC, CAIC, HQIC and K-S test it is evident that the $BKwMO-W$ distribution is a better model than both the $KwMO-W$ and $BMO-W$ for all the data sets considered here. The plots of observed and the estimated pdfs and cdfs also support the same findings. The likelihood ratio test has favored the $BKwMO-W$ in four out of the six comparisons considered here.

CONCLUSION

A new Beta Kumaraswamy Marshall-Olkin family of distributions is introduced and some of its important properties are studied. The maximum likelihood method for estimating the parameters is implemented. Three applications of real life data fitting shows encouraging result in favour of the proposed unified family over Kumaraswamy Marshall-Olkin and the Beta Marshall-Olkin family of distributions.

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REFERENCES

Alizadeh, M., Tahir, M.H., Cordeiro, G.M., Zubair, M. & Hamedani, G.G. (2015a). The beta Marshall-Olkin family of distributions. *Journal of Statistical Distributions and Applications* 2:4 doi:10.1186/s40488-015-0027-7.

Alizadeh, M., Tahir M.H., Cordeiro G.M., Zubair M. & Hamedani G.G. (2015b). The Kumaraswamy Marshal-Olkin family of distributions. *Journal of the Egyptian Mathematical Society* 23: 546-557.

Aarset, M.V. (1987). How to identify bathtub hazard rate. *IEEE Transactions on Reliability* 36: 106-108.

Bekker, A., Roux, J. & Mostert, P. (2000). A generalization of the compound Rayleigh distribution: using a Bayesian methods on cancer survival times. *Commun. Stat. Theory Methods* 29: 1419–1433.

Chakraborty, S. & Handique, L. (2017a). The Generalized Marshall-Olkin-Kumaraswamy-G family of distributions. *Journal of Data Science* 15: 391-422.

Chakraborty, S. & Handique, L. (2017b). Properties and data modeling applications of the Kumaraswamy-Generalized Marshall-Olkin-G family of distributions. (under review).

Cordeiro, G.M. & De Castro, M. (2011). A new family of generalized distributions. *J Stat Comput Simul* 81: 883-893.

Eugene, N., Lee, C. & Famoye, F. (2002). Beta-normal distribution and its applications. *Commun. Stat. Theory Methods* 3: 497-512.

Greenwood, J.A., Landwehr, J.M., Matalas, N.C. & Wallis, J.R. (1979). Probability weighted moments: definition and relation to parameters of several distributions expressible in inverse form. *Water Resour Res* 15:1049-1054.

Handique, L. & Chakraborty S. (2016). The Beta Generalized Marshall-Olkin-G family of distributions. arXiv:1608.05985 V1 [math.ST]. (under review)

Handique, L., Chakraborty, S. & Hamedani, G.G. (2017a). The Marshall-Olkin-Kumaraswamy-G family of distributions. *Journal of Statistical Theory and Applications* (forthcoming, accepted for publication).

Handique, L., Chakraborty, S. & Ali, M.M. (2017b). The Beta-Kumaraswamy-G family of distributions. *Pakistan Journal of Statistics* 33: 467-490.

Jones, M.C. (2004). Families of distributions arising from the distributions of order statistics. *Test* 13: 1-43.

Kenney, J.F. & Keeping, E.S. (1962). Mathematics of Statistics, Part 1. Van Nostrand, New Jersey, 3rd edition.

Krishna, E., Jose, K.K. & Ristić, M. (2013). Applications of Marshall-Olkin Fréchet distribution. *Comm. Stat. Simulat. Comput* 42: 76-89.

Lomax, K.S. (1954). Business Failures; Another example of the analysis of failure data. *Journal of the American Statistical Association* 49: 847-852.

Marshall, A. & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. *Biometrika* 84: 641-652.

Moors, J.J.A. (1988): A quantile alternative for kurtosis. The *Statistician* 37: 25–32.

Nadarajah, S., Cordeiro, G.M. & Ortega, E.M.M. (2015). The Zografos-Balakrishnan–G family of distributions:mathematical properties and applications. *Commun. Stat. Theory Methods* 44: 186-215.

Smith, R.L. & Naylor, J.C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution**.** *Applied Statistics* 36: 358-369.

Song, K.S. (2001). Rényi information, loglikelihood and an intrinsic distribution measure. *Journal of Statistical Planning and Inference* 93:51-69.

Weibull, W. (1951). A statistical distribution function of wide applicability. *J. Appl. Mech.-Trans. ASME*, 18: 293-297.